# Fully automated design of compensated multivariable linear systems

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**Abstract:** The current paper establishes a rapid, real-time algorithm for the synthesis of the reaction matrices used by compensators that reduces multivariable linear systems to extended systems that are cyclical. Thus, the driving of the extended systems (original multivariable system plus compensator) is reduced to a reaction that depends on the outputs and a set of references calculated by the proposed algorithm. The algorithm derives the structure of the compensator and the values of its components (amplifiers and integrators) as well as the values of the references. The paper is divided into two main sections: the derivation of the algorithm and the description of the program.

**Keywords:** multivariable linear system, compensator design, automated tools.

# Proiectarea automată a sistemelor lineare multivariabile compensate

**Rezumat:** Lucrarea de față stabilește un algoritm rapid, în timp real pentru sinteza matricilor de reacție folosite de compensatoarele de stare. Această metoda reduce sistemele liniare multivariabile la sisteme extinse care sunt ciclice. Conducerea sistemelor extinse (formate din uniunea sistemului original plus compensatorul de stare) este redusă la o reacție care depinde de ieșiri și de niște referințe calculate de algoritmul propus. Algoritmul propus obține structura compensatorului, valorile componentelor (amplificatoare și integratoare) și valorile intrărilor de referință.

**Cuvinte cheie:** sisteme lineare multivariabile, proiectarea compensatoarelor, programe pentru proiectare automată.

## 1. Introduction

The control of multivariable linear systems has been greatly improved with the introduction of the compensator based design [Brasch, Davison]. The main challenges in designing such systems can be divided into several categories:

- establish the minimal structure of the compensator (the minimal number of amplifiers and integrators needed);
- establish the optimal values of the amplifiers;
- establish the values of the external references driving the extended system;
- perform the above calculations in real time for systems that vary in time;
- convert the theory of the algorithms into a unitary, minimalistic program.

In the following, we address all the above issues. Compensator based design is far superior to estimator based design due to the fact that it is less complex and cheaper [Bernat]. This paper is dedicated to my mentor, prof. dr. eng. Vlad Ionescu, the person responsible for my education in the theory of automatic control.

## 2. Mathematical foundations

**Theorem [**Gelfand**]:** The polynomials $p(s) = p_0 + p_1 s + ... p_n s^n$ and $q(s) = q_0 + q_1 s + ... q_m s^m$ are co-prime if and only if the rank of the "resultant" matrix $R$ is $rank(R) = m + n$ where:

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$$R = \begin{bmatrix}
p_n & 0 & . & . & . & . & . & . & . & . & . & 0 & q_m \\
p_{n-1} & p_n & 0 & . & . & . & . & . & . & . & 0 & q_m & q_{m-1} \\
. & & . & 0 & . & . & . & . & . & 0 & . & q_{m-1} & . \\
. & & & . & 0 & . & . & . & 0 & . & & & . \\
p_{n-m+1} & . & . & . & p_n & 0 & . & 0 & q_m & . & . & . & q_1 \\
p_{n-m} & . & . & . & p_{n-1} & 0 & . & q_m & . & . & . & . & q_0 \\
. & & & & . & 0 & q_m & . & . & . & . & q_0 & 0 \\
p_1 & . & . & . & p_m & q_m & . & . & . & . & q_0 & 0 & 0 \\
p_0 & . & . & . & p_{m-1} & q_{m-1} & . & . & . & q_0 & 0 & . & 0 \\
0 & p_0 & & & p_{m-2} & q_{m-2} & & & q_0 & 0 & . & . & 0 \\
. & 0 & . & & . & . & & . & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & 0 & . & . & . & . & 0 \\
0 & . & . & 0 & p_0 & q_0 & 0 & . & . & . & . & . & 0
\end{bmatrix} \tag{2.1}$$

The subroutine PRIM(R,Q,N,IM,IO) constructs the resultant matrix $R(NM, NM)$ where $NM = N + M, M = N - K$ and $K$ is the order of differentiation of the characteristic polynomial $\mathrm{X}_Q(s) = s^N + \alpha_{N-1}s^{N-1} + ....\alpha_0$ . $\mathrm{X}_Q(s)$ is the characteristic polynomial of matrix $Q$. In other words, PRIM (described later in the paper) constructs the resultant matrix between $\mathrm{X}_Q(s)$ and $\mathrm{X}_Q^{(K)}(s)$ whereby polynomial $\mathrm{X}_Q^{(K)}(s)$ is the K-th derivative of the polynomial $\mathrm{X}_Q(s)$. The subroutine accepts as input the matrix $Q(N+1, N)$ constructed by subroutine CQ as well as the numbers $N, NM = N + (N - K)$ and $IP = K$ , the order of differentiation of the characteristic polynomial and returns the resultant matrix $R(NM, NM)$.

**The Penrose Pseudoinverse Theorem [**Penrose**]:** The system of linear equations:

$$E * X = \Delta \tag{2.2}$$

where $E$ is a $n \times r$ matrix, $X(r)$ and $\Delta(n)$ are vectors has the unique solution of minimal norm:

$$X = E^{\#} * \Delta \tag{2.3}$$

where $E^{\#}$ is the $r \times n$ Penrose pseudoinverse of matrix $E$.

**Brasch-Pearson Theorem I [**Brasch**]**: If the triplet of matrices $(A, B, C)$ whereby matrices $(A, B, C)$ are defined as $A = A(n,n), B = B(n,r), C = C(m,n)$ , is canonical then it exists a matrix $K$ so the matrix $A + BKC$ is cyclical. A matrix is cyclical if all the eigenvalues of its characteristic polynomial have degree of multiplicity 1.

**Brasch-Pearson Theorem II [**Brasch**]**: If the triplet $(A, B, C)$ is canonical then for any set $\Lambda_e = \{\lambda_1, ...\lambda_{n+l}\}$ where $l = \min(p_c, p_o)$ where $p_c, p_o$ are respectively the controllability and observability orders of the system defined by $(A, B, C)$ there exists a matrix $K$ so that the eigenvalues of matrix $A_e + B_e K C_e$ are the elements of $\Lambda_e$ . $A_e = \begin{bmatrix} A & 0_{n,l} \\ 0_{l,n} & 0_{l,l} \end{bmatrix}, B_e = \begin{bmatrix} B & 0_{n,l} \\ 0_{l,r} & I_{l,l} \end{bmatrix}, C_e = \begin{bmatrix} C & 0_{m,l} \\ 0_{l,n} & I_{l,l} \end{bmatrix}$

**Davison-Wang Theorem I [**Davison**]**: If the triplet $(A, B, C)$ is canonical then it exists $K$ so the matrix $A + BKC$ has all distinct eigenvalues (have degree of multiplicity 1).

**Davison-Wang Theorem II [**Davison**]**: If the triplet $(A, B, C)$ is canonical then the set $\mathrm{K} = \{\mathrm{K} | \mathrm{K} \in \mathrm{R}^{\mathrm{r \times m}}, A + BKC$ has some non-distinct eigenvalues$\}$ is either void or is a hypersurface in $\mathrm{R}^{\mathrm{r \times m}}$

**Davison-Wang Theorem III [**Davison**]**: If the triplet $(A,B,C)$ is canonical and $A$ is cyclical then:

- the set $\Lambda = \{\mathrm{l} | \mathrm{l} \in \mathrm{R}^{\mathrm{r}\times 1}, (A, Bl) \neq \text{completely controllable}\}$ is either void or a hypersurface in $\mathrm{R}^{\mathrm{r}\times 1}$
- the set $\mathrm{R} = \{\mathrm{r} | \mathrm{r} \in \mathrm{R}^{1\times \mathrm{m}}, (A, rC) \neq \text{completely observable}\}$ is either void or a hypersurface in $\mathrm{R}^{1\times \mathrm{m}}$

Theorem II Davison-Wang shows that "almost any $K \in R^{r\times m}$ makes $A+BKC$ cyclical. Theorem III Davison-Wang shows that "almost any $l \in R^{r\times 1}$ and almost any $r \in R^{1\times m}$" will make the triplet $(A+BKC, Bl, rC)$ canonical where $K \in R^{r\times m}$ is the one that made $A+BKC$ cyclical.

## 3. Compensator-assisted pole allocation

Let the dynamics of a system be described by the equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.1}$$

where $A(n,n), B(n,r), C(m,n)$ are time invariant. Let $u = Ky + w$ where $w$ are some references to be determined later. Then, the system becomes:

$$\begin{aligned} \dot{x} &= (A+BKC)x + Bw \\ y &= Cx \end{aligned} \tag{3.2}$$

The transfer function is:

$$G(s) = C(sI_n - A - BKC)^{-1}B \tag{3.3}$$

Let: $A_e = \begin{bmatrix} A & 0_{n,l} \\ 0_{l,n} & 0_{l,l} \end{bmatrix}, B_e = \begin{bmatrix} B & 0_{n,l} \\ 0_{l,r} & I_{l,l} \end{bmatrix}, C_e = \begin{bmatrix} C & 0_{m,l} \\ 0_{l,n} & I_{l,l} \end{bmatrix}$

The first index represents the number of lines in the matrix, the second index represents the number of columns. Let $p_c, p_o$ be the smallest positive integers for which:

$$\mathrm{rank}[B, AB, ....A^{p_c}B] = n, \mathrm{rank}[C^T, A^T C^T, ....(A^T)^{p_o} C^T] = n \tag{3.4}$$

The, the second Brash-Pearson theorem tells us that:

If the triplet $(A,B,C)$ is canonical then for any set $\Lambda_e = \{\lambda_1, ...\lambda_{n+l}\}$ where $l = \min(p_c, p_o)$ there exists a matrix $K$ so that the eigenvalues of matrix $A_e + B_e K C_e$ are the elements of $\Lambda_e$.

Proof: according to Theorem I Davison-Wang if the triplet $(A,B,C)$ is canonical then it exists $K_1$ so the matrix $A_1 = A + BK_1C$ has all distinct eigenvalues (is cyclical). Obviously $(A_1, B, C)$ is canonical. Let:

$$K = \begin{bmatrix} K_1 & 0_{r,l} \\ 0_{l,m} & 0_{l,l} \end{bmatrix} + K_2 \tag{3.5}$$

$$A_e + B_e K C_e = A_e^1 + B_e K_2 C_e \text{ where } A_e^1 = \begin{bmatrix} A + BK_1C & 0_{n,l} \\ 0_{l,n} & 0_{l,l} \end{bmatrix} \tag{3.6}$$

So, all that is left to prove is that there exists $K_2$ so that the $A_e^1 + B_e K_2 C_e$ eigenvalues are the elements of $\Lambda_e$. The transfer function of the extended system is:

$$G_e(s) = C_e(sI_{n+l} - A_e^1 - B_e K_2 C_e)^{-1} B_e \tag{3.7}$$

The poles of the extended system are given either by the equation:

$$\Delta_e(s) = \frac{\det\{sD(s)I_{r+l} - K_2 C_e \begin{bmatrix} sD(s)(sI - A_1)^{-1} & 0_{n,l} \\ 0_{l,n} & D(s)I_l \end{bmatrix} B_e\}}{s^r D^{r+l-1}(s)} = 0 \tag{3.8}$$

or by the equation:

$$\Delta_e(s) = \frac{\det\{sD(s)I_{m+l} - C_e \begin{bmatrix} sD(s)(sI - A_1)^{-1} & 0_{n,l} \\ 0_{l,n} & D(s)I_l \end{bmatrix} B_e K_2\}}{s^m D^{m+l-1}(s)} = 0 \tag{3.9}$$

where:

$$D(s) = \det(sI - A_1) = s^{n-1} + \alpha_{n-1}^1 s^{n-2} + .... + \alpha_1^1 s + \alpha_0^1 \tag{3.10}$$

On the other hand, the desired characteristic polynomial of the extended matrix $A_e^1 + B_e K_2 C_e$ has the form:

$$\Delta_e(s) = s^{n+l} + \beta_{n+l-1} s^{n+l-1} + ..... + \beta_1 s + \beta_0 \tag{3.11}$$

There are two cases to be studied:

a) $l = p_c$ Since $A_1$ has all distinct eigenvalues (is cyclical) and $(A_1, C)$ is completely observable, it follows that there exists $\eta = \begin{bmatrix} \eta_1 \\ . \\ . \\ \eta_m \end{bmatrix}$ so that $(A_1, g^T)$ is completely observable, where $g = C^T \eta$. Even more than that, “almost any $\eta$” makes $(A_1, g^T)$ completely observable. Let’s choose $K_2$ to be of the form:

$$K_2 = \begin{bmatrix} k_1\eta^T & k_{1,m+1} & . & . & . & . & . & k_{1,m+l} \\ . & . & & & & & & . \\ . & . & & & & & & . \\ k_r\eta^T & k_{r,m+1} & . & . & . & . & . & k_{r,m+l} \\ k_{r+1}\eta^T & k_{r+1,m+1} & . & . & . & . & . & k_{r+1,m+l} \\ . & . & & & & & & . \\ . & . & & & & & & . \\ k_{r+l}\eta^T & k_{r+l,m+1} & . & . & . & . & . & k_{r+l,m+l} \end{bmatrix} \tag{3.12}$$

We can further refine the above form for $K_2$ to:

$$K_2 = \begin{bmatrix} (\delta_1^0-\delta_1^1)\eta^T & \delta_2^0-\delta_2^1-\delta_1^0(\delta_1^0-\delta_1^1) & . & . & . & . & . & \delta_{l+1}^0-\delta_{l+1}^1-\delta_l^0(\delta_1^0-\delta_1^1) \\ . & . & & & & & & . \\ . & . & & & & & & . \\ (\delta_1^0-\delta_r^1)\eta^T & \delta_2^0-\delta_2^r-\delta_1^0(\delta_1^0-\delta_1^r) & . & . & . & . & . & \delta_{l+1}^0-\delta_{l+1}^r-\delta_l^0(\delta_1^0-\delta_1^r) \\ \eta^T & -\delta_1^0 & . & . & . & . & . & -\delta_1^l \\ 0 & 1 & 0 & . & . & . & . & 0 \\ . & . & 1 & 0 & . & . & . & . \\ 0 & k_{r+l,m+1} & . & . & . & . & 1 & 0 \end{bmatrix} \tag{3.13}$$

where $\delta_{l+1}^0 = 0$. Substituting the expression for $K_2$, we obtain:

$$\Delta_e(s) = D(s)s^l + \sum_{i=1}^{l+1}[\delta_i^0 D(s)s^{l-i} + s^{l+1-i}\sum_{j=1}^{r}(\delta_i^j - \delta_i^0)N_j(s)] \tag{3.14}$$

In the above:

$$[N_1(s),.....,N_r(s)] = [s^{n-2}, s^{n-1},...1]*L \tag{3.15}$$

$$L^T = [g^T B, g^T A_1 B + \alpha_{n-1}^1 g^T B,....,g^T A_1^{n-1}B + \sum_{j=1}^{n-1} \alpha_{n-j}^1 g^T A_1^{n-j-1}B] \tag{3.16}$$

Requiring that the two characteristic polynomials are identical we obtain the system:

$$\begin{bmatrix} 1 & 0 & . & . & . & . & . & . & . & 0 & 0 & \uparrow \\ \alpha_{n-1}^1 & 1 & & & & & & & & 0 & \uparrow & \\ . & \alpha_{n-1}^1 & . & & & & & & & \uparrow & & L \\ . & & . & . & & & & & \uparrow & & L & \\ . & & & \alpha_{n-1}^1 & 1 & & & & & L & & \\ . & & & & \alpha_{n-1}^1 & & \uparrow & & L & & & \\ . & \alpha_1^1 & & & & \uparrow & & . & & & & \downarrow \\ . & . & . & & & & . & & & & \downarrow & 0 \\ \alpha_1^1 & . & . & . & & L & & & & \downarrow & 0 & . \\ \alpha_0^1 & . & . & . & . & & & & \downarrow & 0 & & . \\ 0 & . & . & . & . & & & & 0 & & & . \\ . & . & & & & & & . & & & & . \\ . & & . & \alpha_0^1 & \alpha_1^1 & & \downarrow & & & & & . \\ 0 & . & . & 0 & \alpha_0^1 & \downarrow & 0 & & & & & 0 \end{bmatrix} * \begin{bmatrix} \delta_1^0 \\ . \\ . \\ \delta_l^0 \\ \delta_{l+1}^1 - \delta_{l+1}^0 \\ . \\ . \\ \delta_{l+1}^r - \delta_{l+1}^0 \\ . \\ . \\ \delta_1^1 - \delta_1^0 \\ . \\ . \\ \delta_1^r - \delta_1^0 \end{bmatrix} = \begin{bmatrix} \beta_{n+l-1} - \alpha_{n-1}^1 \\ . \\ . \\ . \\ . \\ . \\ \beta_l - \alpha_0^1 \\ . \\ . \\ \\ \\ \\ \\ \beta_0 \end{bmatrix} \tag{3.17}$$

The above can be written in the contracted form:

$$E * X = \Delta \tag{3.18}$$

In (3.18) the matrix $E$ has dimensions $(l+n)\times[l+r(l+1)]$, $X$ has dimension $[l+r(l+1)]\times 1$ and $\Delta$. Is the vector of dimension $n+l$. The arrows in matrix (3.17) are a marker for the extent of vector $L$, i.e. they show the number of rows occupied by the vector $L$. At this point, in order to complete the proof of Brash-Pearson theorem, we need to introduce the following:

**Theorem:** The system (3.18) is compatible and $\text{rank}(E) = n+l$.

Proof: Since $\text{rank}[B, AB,....A^{p_c}B] = n$ it follows that $r(l+1) \geq n$, so $l + r(l+1) \geq l+n$ therefore the system has more unknowns that equations. If we can show that $\text{rank}(E) = n+l$ then the system is compatible, so there exists at least one solution. In order to do that we need to bring

$E$ to the block diagonal form: $$E \sim \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ x & . & & & & . \\ . & & . & & & . \\ x & . & x & 1 & 0 & 0 \\ . & & . & & S_{n,r(l+1)} & \\ . & & . & & & \end{bmatrix} \tag{3.19}$$

where the upper left corner block is a $l \times l$ matrix and the lower right corner block is a $n \times r(l+1)$ matrix. So, all we need to show is that $\operatorname{rank}(S) = n$. It is easy to show that:

$$S \sim \begin{bmatrix} g^T \\ g^T A_1 \\ . \\ . \\ g^T A_1^{n-1} \end{bmatrix} * [B, AB, ...., A_1^l B] \tag{3.20}$$

We know that:

$\operatorname{rank}[g, A_1 g, ....A_1^{n-1} g] = n$ because $g$ was determined such that $(A_1, g)$ is completely observable

$\operatorname{rank}[B, AB, ...., A_1^l B] = n$ because $l = p_c$

Therefore:

$$\begin{aligned} &\operatorname{rank}(S) = n \\ &\operatorname{rank}(E) = n + l \end{aligned} \tag{3.21}$$

If $l + r(l+1) = l + n$ then the system is completely determined and its solution is $X = E^{-1} * \Delta$

If $l + r(l+1) > l + n$ then the system undetermined and there is at least one solution, the solution of minimum norm is $X = E^{\#} * \Delta$ where $E^{\#}$ is the Penrose pseudoinverse of matrix $E$. Once we have determined $X$ $K_2$ follows immediately. Following Theorem I Davison-Wang we have determined earlier matrix $K_1$ that makes $A_1 = A + BK_1C$ cyclical, so, the reaction matrix that gives the desired pole allocation for the extended system is:

$$K = \begin{bmatrix} K_1 & 0_{r,l} \\ 0_{l,m} & 0_{l,l} \end{bmatrix} + K_2 \tag{3.22}$$

In a later section we will show a very elegant method of computing the additional references $w$.

$l = p_o$

Since $A_1$ has all distinct eigenvalues (is cyclical) and $(A_1, B)$ is completely controllable, it follows that there exists $h = \begin{bmatrix} h_1 \\ . \\ . \\ h_r \end{bmatrix}$ so that $(A_1, t)$ is completely controllable, where $t = Bh$. Even more than that, “almost any $h$” makes $(A_1, t)$ completely observable. Let’s choose $K_2$ to be of the form:

$$K_2 = \begin{bmatrix} (\delta_1^0-\delta_1^1)h & \cdot & \cdot & (\delta_1^0-\delta_1^m)h & h & 0 & \cdot & \cdot & 0 \\ \delta_2^0-\delta_2^1-\delta_1^0(\delta_1^0-\delta_1^1) & \cdot & \cdot & \delta_2^0-\delta_2^m-\delta_1^0(\delta_1^0-\delta_1^m) & -\delta_1^0 & & & & -\delta_l^0 \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & & & \cdot & 0 & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \delta_{l+1}^0-\delta_{l+1}^1-\delta_l^0(\delta_1^0-\delta_1^1) & \cdot & \cdot & \delta_{l+1}^0-\delta_{l+1}^m-\delta_1^0(\delta_l^0-\delta_1^m) & 0 & \cdot & 0 & 1 & 0 \end{bmatrix} \tag{3.23}$$

Let:

$$L^T = [Ct, CA_1t+\alpha_{n-1}^1 Ct, ...., CA_1^{n-1}t + \sum_{j=1}^{n-1} \alpha_{n-j}^1 CA_1^{n-j-1}t] \tag{3.24}$$

Following the same reasoning as the one used at point a) we obtain the system:

$$\begin{bmatrix} 1 & & & & & & & & & & & \uparrow \\ \alpha_{n-1}^1 & 1 & & & & & & & & & \uparrow & \\ \cdot & \alpha_{n-1}^1 & \cdot & & & & & & & \uparrow & & L \\ \cdot & & \cdot & \cdot & & & & & \uparrow & & L & \\ \cdot & & & \alpha_{n-1}^1 & 1 & & & & & L & & \\ \cdot & & & & \alpha_{n-1}^1 & & \uparrow & & L & & & \\ \cdot & \alpha_1^1 & & & & \uparrow & & \cdot & & & & \downarrow \\ \cdot & \cdot & \cdot & & & & \cdot & & & & \downarrow & \\ \alpha_1^1 & \cdot & \cdot & \cdot & & L & & & & \downarrow & & \\ \alpha_0^1 & \cdot & \cdot & \cdot & \cdot & & & & \downarrow & & & \\ 0 & \cdot & \cdot & \cdot & \cdot & & & & & & & \\ \cdot & \cdot & & & & & & & & & & \\ \cdot & & \cdot & \alpha_0^1 & \alpha_1^1 & & & & & & & \\ 0 & \cdot & \cdot & 0 & \alpha_0^1 & \downarrow & & & & & & \end{bmatrix} * \begin{bmatrix} \delta_1^0 \\ \cdot \\ \cdot \\ \delta_l^0 \\ \delta_{l+1}^1-\delta_{l+1}^0 \\ \cdot \\ \cdot \\ \delta_{l+1}^m-\delta_{l+1}^0 \\ \cdot \\ \cdot \\ \delta_1^1-\delta_1^0 \\ \cdot \\ \cdot \\ \delta_1^m-\delta_1^0 \end{bmatrix} = \begin{bmatrix} \beta_{n+l-1}-\alpha_{n-1}^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_l-\alpha_0^1 \\ \cdot \\ \cdot \\ \\ \\ \\ \\ \beta_0 \end{bmatrix} \tag{3.25}$$

The above can be written in the contracted form:

$$F * X = \Delta \tag{3.26}$$

Exactly as at point a) we show that $\text{rank}(F) = n+l$ and the solution of the system is $X = F^{\#} * D$. $F$ has dimension $(l+n)\times(l+m(l+1))$, $X$ is a vector with dimension $l+m(l+1)$ and $\Delta$ is a vector with dimension $n+l$.

## 4. Description of the subroutines

UNIT(B,N) creates the unit matrix $B(N,N) = I_N$

TRAM(A,B,M,N) creates the transpose of matrix $A(M.N)$ : $B(N.M) = A^T(M.N)$

PROMAT(A,B,C,M,L,N) constructs the product: $C(M,N) = A(M,L) * B(L,N)$

HAZ(S,N,Z) creates the random vector $Z(N)$ from an arbitrary seed number $S$

ZADEH(A,N,ALFA) calculates the coefficients $\alpha_0 = ALFA(1), ......., \alpha_{n-1} = ALFA(N)$ of the characteristic polynomial associated with the matrix $A(N,N)$ by using the algorithm described by Zadeh

DIAG(A,B,B,RO,I1,I2) brings the matrix $A(N,N)$ to the diagonal form $B(N,N)$. DIAG returns the rank $RO$ of matrix $A(N,N)$ as well as $I1$, the index of the first non-null diagonal element and $I2$, the index of the last non-null diagonal element, to be used in the calculation of the pseudoinverse (see next).

PS(L,N,R,LPI) constructs the Penrose pseudoinverse of matrix $L(N,R)$ , $LPI(R,N) = L^{\#}(N,R)$

ESALON(A,M,N) constructs the form of reduced echelon of matrix $A(M.N)$

CONOBS(IV,V,W,AA,N) constructs the matrix $W(N, N*IV) = [V, AA*V, ...., AA^{N-1}*V]$

CQ(Q,N,AL) constructs the coefficients of the higher order derivatives of characteristic polynomial $X_A(s) = s^N + \alpha_{N-1}s^{N-1} + ....\alpha_0$ associated with the matrix $A(N,N)$ . These coefficients are stored column by column in the matrix $Q(N+1,N)$. The first column is the coefficients of the characteristic polynomial $[1,\alpha_{N-1},...\alpha_0]$, the last column are the coefficients of the n-th derivative. The matrix $Q(N+1,N)$ is used by PRIM (see next) in order to determine the order of multiplicities of the roots of the characteristic polynomial.

PRIM(R,Q,N,NM,IP) constructs the resultant matrix $R(NM,NM)$ where $NM = N+M, M = N-K$ and $IP = K$ where $K$ is the order of differentiation of the characteristic polynomial . $X_A(s) = s^N + \alpha_{N-1}s^{N-1} + ....\alpha_0$. In other words, PRIM constructs the resultant matrix between $X_A(s)$ and $X_A^{(K)}(s)$ . The subroutine accepts as input the matrix $Q(N+1,N)$ constructed by CQ as well as $N, NM = N+(N-K)$ and $IP = K$, the order of the derivative of the characteristic polynomial and returns the resultant matrix $R(NM,NM)$.

## 5. Description of the main program

Input matrices $A(N,N), B(N,IR), C(M,N)$ describing the dynamics of the system to be compensated.

Construct the controllability matrix $QC(N, N*IR) = (B, AB, A^2B...A^N B)$

Bring $QC$ QC to the echelon reduced form in order to calculate the controllability index IC

Determination of the controllability index IC:

Construct the observability matrix $QO(N, N*M) = (C^T, A^TC^T, (A^T)^2C^T...(A^T)^N C^T)$

Bring $QO$ to the echelon reduced form in order to calculate the controllability index IO

Determination of the observability index IO

Generate matrix $K1(IR,M)$ that makes $Y = A + B*K1*C$ cyclic

$A_1(N,N) = Y(N,N)$

$\det(sI - A_1) = s^N + \alpha_{N-1}^1 s^{N-1} + .... + \alpha_1^1 s + \alpha_0^1$

IF we were NOT successful in making $Y = A + B*K1*C$ cyclic, we will apply the algorithm Davison-Wang

IF we failed to bring Y to the cyclic form in our first attempt, we can try a programmable number of more tries

We managed to bring Y to the cyclic form $(A,G)$

Generate the random vector $ETA(N)$ and the random vector $G(N) = C^T(M,N) * ETA(N)$ that makes the pair $(A,G)$ completely observable.

IF we failed to find G in a programmable number of tries, we have to give up or try more times.

Generate the random vector $H(IR)$ and the random vector $T(N) = B(N,IR) * H(IR)$ that makes the pair $(A,T)$ completely controllable.

IF we failed to find T in a programmable number of tries, we have to give up or try more times.

C Determine the number of integrators $L = \min(IC, IO)$.

Input the coefficients of the desired characteristic polynomial in the form of the vector $BETA(N+L)$.

IF L=IC Then

Begin

Construct matrix $CL(N,IR) = \begin{bmatrix} G^T * B \\ G^T * A_1 * B + a_{N-1}^1 G^T * B \\ . \\ . \\ . \end{bmatrix}$

Construct matrix $E(N+L, L+(L+1)IR)$

Construct matrix $D(N+L)$

Construct the pseudoinverse of matrix $E$, the matrix $E^{\#}(L+(L+1)IR, N+L)$

Construct the matrix $X(L+(L+1)IR) = E^{\#}(L+(L+1)IR, N+L) * D(N+L)$

Construct matrices $\begin{matrix} CK1(IR+L, M+L), CK2(IR+L, M+L), \\ CK = CK1 + CK2 \end{matrix}$

The solution is matrix $CK(IR+L, M+L)$

End Else

Begin

Construct matrix $OL(N,M)$

Construct matrix $F(N+L, L+(L+1)M)$

Construct matrix $D(N+L)$

Construct the pseudoinverse of matrix $F$, the matrix $F^{\#}(L+(L+1)M, N+L)$

Construct the matrix $X(L+(L+1)M) = F^{\#}(L+(L+1)IM, N+L) * D(N+L)$

Construct matrices $OK1(IR+L, M+L), OK2(IR+L, M+L), OK = OK1 + OK2$

The solution is matrix $OK(IR+L, M+L)$

End

Algorithm self-check

Form the extended matrices

$AE(N+L, N+L), BE(N+L, IR+L), CE(M+L, N+L), L = \min(IC, IO)$

IF IC<IO find the coefficients $ALFA2(N+L)$ of the characteristic polynomial of matrix $AE + BE*CK*CE$

IF IC>IO find the coefficients $ALFA2(N+L)$ of the characteristic polynomial of matrix $AE + BE*OK*CE$

IF IC=IO, you can choose either $CK$ or $OK$

Compare $ALFA2(N+L)$ with the target coefficients $BETA(N+L)$, they must be equal

Calculate the references: $w_{r,1} = TR(0)^{\#}_{r,m} y_{steady}$ where $TR(0)^{\#}_{r,m}$ is the pseudoinverse of $TR(0)_{r,m}$ and:

$$TR(0) = C'\frac{1}{a^1_{0e}}[\alpha^1_{1e}I + .. + \alpha^1_{n-1,e}(A^1_e)^{n-2} + (A^1_e)^{n-1}]B'$$
$$n = N + L$$
$$C' = \begin{bmatrix} C_{m,N} & 0_{m,L} \end{bmatrix}, B' = \begin{bmatrix} B_{N,r} \\ 0_{L,r} \end{bmatrix}$$
$$m = M, r = IR$$

The "$e$" subscript stands for "extended system" (system including the compensator). The dynamic dimensions of the matrices used by the program are:

$A(N,N), B(N,IR), C(M,N), BETA(N+L),$
$AE(N+L, N+L), BE(N+L, IR+L),$
$CE(M+L, N+L), BE1(N+L, M+L),$
$CE1(N+L, N+L), A2(N+L, N+L),$
$ALFA2(N+L), ACK(IR, N), BAKC(N, N),$
$Y(N,N), QC(N, N*IR), QO(N, N*M),$
$T(N), G(N), R(N,N), E(N+L, L+(L+1)IR),$
$PSE(L+(L+1)IR, N+L),$
$F(N+L, L+(L+1)IM), PSF(L+(L+1)M, N+L),$
$P(N,N), CK1(IR+L, M+L),$
$CK2(IR+L, M+L), CK(IR+L, M+L),$
$OK1(IR+L, M+L), OK2(IR+L, M+L),$
$OK(IR+L, M+L), D(N+L),$
$X(\max\{L+(L+1)IR, L+(L+1)M\}), H(IR), ETA(M),$
$CL(N, IR), OL(N, M), AK1(IR, M),$
$AS(IR*M), AK(IR, M)$

The program can be downloaded from link [5].

## 6. A practical example – a system with 3 inputs and 2 outputs

Consider the system described by the equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6.1}$$

where $A, B, C$ are time invariant:

$$A=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix},B=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix},C=\begin{bmatrix}1&0&0&0&0\\0&0&0&1&0\end{bmatrix}.N=5, IR=3, M=2 \tag{6.2}$$

The controllability index is found to be $IC=1$ and the observability index is found to be $IO=2$ so, $L=\min(IC,IO)=1$ .Therefore, the number of compensator integrator blocks is one. The rank of the extended system is $N+L=6$ so we have to allocate six poles for the extended system. We do that by specifying a characteristic polynomial of order six: $X_{A_f}(s)=s^6-2s^5+4s^4+s^3-3s^2-5s+2$ . We input the coefficients of the desired characteristic polynomial in the form of the vector $BETA(N+L)=[-2,4,1,-3,-5,2]$. The program will furnish the feedback matrix: $CK(IR+L,M+L)=\begin{bmatrix}-7.44&-2.66&0.02\\0.85&-8.88&-4.23\\12.14&1.06&14.6\\-2.58&-15.7&-2\end{bmatrix}$ (6.3)

The dynamic compensator consists of: 12 amplifiers (see the elements of matrix $CK$ ), 4 summing units with 4 inputs each (see diagram) and 1 integrator. Note that the compensated system has acquired and extra input, $u_4$ and an extra output, $y_3$. If the values for the amplifiers are inconvenient, it is very easy to generate another matrix $CK$ by generating a new pair of random vectors $ETA(N)$, $H(IR)$.

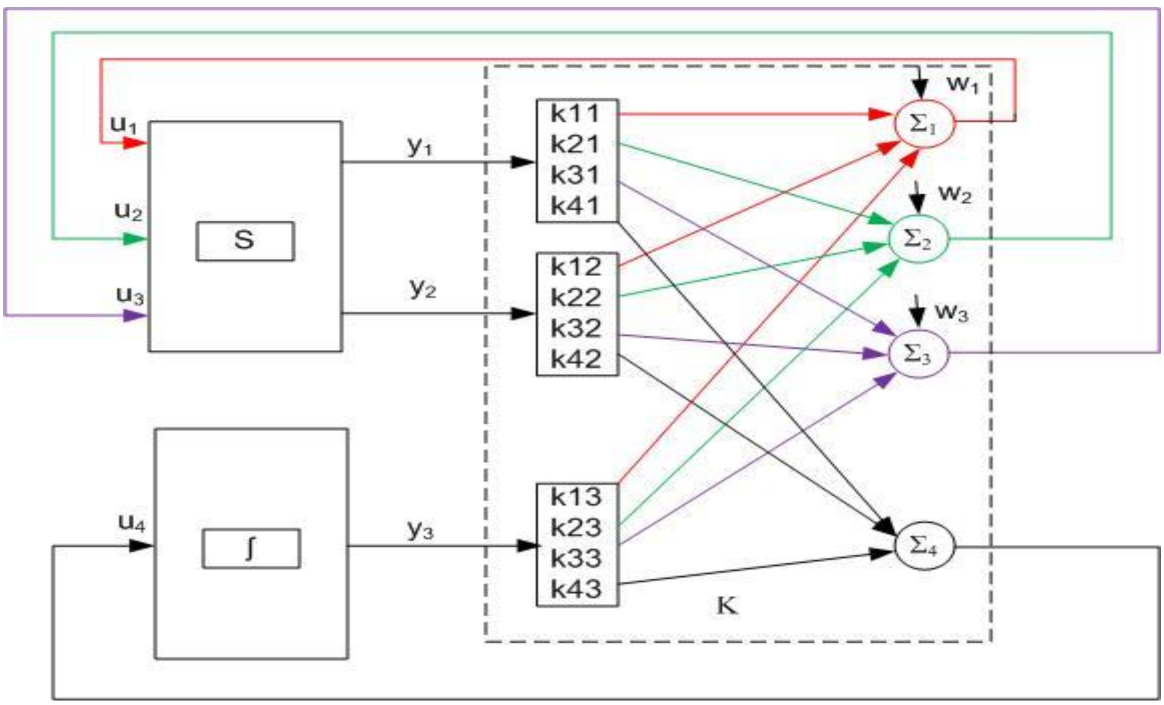


**Figure 1.** The diagram of the compensated system

We are left with the task of computing the references for the compensator: $w_1, w_2, w_3$. We will give the general algorithm for a very simple calculation of the compensator references. The dynamics of the compensated system are described by:

$$\dot{x}_l = A_e x_e + B_e u_e$$
$$y_e = C_e x_e$$
$$A_e = \begin{bmatrix} A & 0_{n,l} \\ 0_{l,n} & 0_{l,l} \end{bmatrix}, B_e = \begin{bmatrix} B & 0_{n,l} \\ 0_{l,r} & I_{l,l} \end{bmatrix}, C_e = \begin{bmatrix} C & 0_{m,l} \\ 0_{l,n} & I_{l,l} \end{bmatrix}, x_e = \begin{bmatrix} x_{n,1} \\ \bar{x}_{l,1} \end{bmatrix}, y_e = \begin{bmatrix} y_{m,1} \\ \bar{y}_{l,1} \end{bmatrix}, u_e = \begin{bmatrix} u_{r,1} \\ \bar{u}_{l,1} \end{bmatrix} \quad (6.4)$$
$$l = L, m = M, r = IR, n = N$$

The "$e$" subscript stands for "extended system" (system including the compensator). The above system splits into two separate systems:

$$\dot{x} = Ax + Bu$$
$$y = Cx \quad (6.5)$$

$$\dot{\bar{x}} = I_{l,l} u$$
$$\bar{y} = I_{l,l} \bar{x} \quad (6.6)$$

According to Theorem II Brasch-Pearson, there exists $K$ such that the extended system obtained by using the output feedback $u_l = Ky_l + w_l$ will have the dynamics described by:

$$\dot{x}_e = (A_e + B_e K C_e) x_e + B_e w_e$$
$$y_e = C_e x_e \quad (6.7)$$

and the matrix $A_e^1 = A_e + B_e K C_e$ will have the desired distribution of the eigenvalues and $w_e = \begin{bmatrix} w_{r,1} \\ 0_{l,1} \end{bmatrix}$

$$\dot{x}_e = A_e^1 x_e + B_e w_e$$
$$y_e = C_e x_e \quad (6.8)$$

This means that:

$$Y_e(s) = C_e (sI_{n+l} - A_e^1)^{-1} B_e W(s) = T_{m+l,r+l}(s) * W_{r+l,1}(s) \quad (6.9)$$

Because $W(s) = \begin{bmatrix} W_{r,1}(s) \\ 0_{l,1} \end{bmatrix}$ and $Y(s) = \begin{bmatrix} Y_{m,1}(s) \\ \bar{Y}_{l,1}(s) \end{bmatrix}$ we can write:

$$Y_{m,1}(s) = TR_{m,r}(s) * W_{r,1}(s) \quad (6.10)$$

The steady state of the outputs is:

$$Y_{steady}(s) = Y(\infty) = \lim_{s \to 0} sY(s) = TR_{m,r}(0) \lim_{s \to 0} sW_{r,1}(s) = TR_{m,r}(0) W_{steady} \quad (6.11)$$

It follows that:

$$y_{steady} = TR(0) * w \quad (6.12)$$

So, the desired vector of constant references is:

$$w_{r,1} = TR(0)^{\#}_{r,m} * y_{steady} \tag{6.13}$$

Now, we remember that:

$$(sI - A)^{-1} = \frac{1}{X_A(s)}[(s^{n-1} + \alpha_{n-1}s^{n-2} + .... + \alpha_1)I + .. + (s + \alpha_{n-1})A^{n-2} + A^{n-1}] \tag{6.14}$$

It follows that:

$$(sI_{n+l,n+l} - A_e^1)^{-1}|_{s=0} = \frac{1}{a_{0,e}^1}[\alpha_{1,e}^1 I + .. + \alpha_{n-1,e}^1 (A_e^1)^{n-2} + (A_e^1)^{n-1}] \tag{6.15}$$

$$T(0) = C_e \frac{1}{a_{0,e}^1}[\alpha_{1,e}^1 I + .. + \alpha_{n-1,e}^1 (A_e^1)^{n-2} + (A_e^1)^{n-1}]B_e \tag{6.16}$$

$$TR(0) = C' \frac{1}{a_{0,e}^1}[\alpha_{1,e}^1 I + .. + \alpha_{n-1,e}^1 (A_e^1)^{n-2} + (A_e^1)^{n-1}]B'$$

$$C' = \begin{bmatrix} C_{m,N} & 0_{m,L} \end{bmatrix}, B' = \begin{bmatrix} B_{N,r} \\ 0_{L,r} \end{bmatrix}, n = N + L \tag{6.17}$$

This way, the algorithm avoids the calculation of the transfer matrix $T(s)$ altogether. The program furnishes the matrix $A_e^1$ (denoted as $A2(N+L, N+L)$ in the program) and the coefficients $\alpha_{i,e}^1$ of the characteristic polynomial $X_{A_e^1}(s)$ , denoted as $ALFA2(N+L)$ in the program. The algorithm has the added strength that if the desired steady outputs $y_{steady}$ vary, then the only thing needed in re-calculating the references is to re-calculate the product $TR(0)^{\#}_{r,m} y_{steady}$, $TR(0)^{\#}_{r,m}$ being time invariable. The author is grateful for the help of Sreeganesh and Sreegurunath Siva in debugging the program. The author also thanks the anonymous referee for the clarifications added to the original text.

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