Chaos theory, a modern approach of nonlinear dynamic systems

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Abstract: Chaos theory is a branch of mathematics focusing on nonlinear dynamic systems. As a relatively new field with a significant applicability area, chaos theory is an active research area involving many different disciplines (mathematics, topology, physics, social systems, population modeling, biology, meteorology, astrophysics, information theory, computational neuroscience, cryptography, robotics etc.). The availability of cheaper and more powerful computers has made a major contribution to the achievement of major advances in nonlinear dynamic systems theory, the interest in deterministic chaos has increased enormously, being reflected in both literature and real life.

Keywords: *chaos, determinism, dynamical system, phase space, attractor, fractal.*

Teoria haosului, o abordare modernă a sistemelor dinamice neliniare

Rezumat: Teoria haosului este o ramură a matematicii care se ocupă de sisteme dinamice neliniare. Domeniu relativ nou, cu o plajă semnificativă de aplicabilitate, teoria haosului reprezintă o zonă activă de cercetare, care implică multe discipline diferite (matematică, topologie, fizică, sisteme sociale, modelare populațională, biologie, meteorologie, astrofizică, teoria informațiilor, neuroștiințe computaționale, criptografie, robotică etc.). Disponibilitatea calculatoarelor mai ieftine și mai puternice a contribuit esențial la înregistrarea de progrese majore în teoria sistemelor dinamice neliniare, interesul pentru haosul determinist a crescut enorm, aspect reflectat atât în literatura de specialitate, cât și în viața reală.

Cuvinte cheie: *haos, determinism, sistem dinamic, spațiul fazelor, atractor, fractal.*

Introduction

Most data analysis methods use linear models that are based on relationships described by linear differential equations because they are easy to manipulate and usually give unique solutions. However, nonlinear behavior occurs frequently in real-life systems due to their complex dynamic nature. This cannot be adequately described by linear models; the use of chaotic systems is an appropriate solution.

A *system* is a set of components that interact and form a whole; *nonlinearity* means that due to feedback or multiplicative effects between components, the whole becomes more than the sum of the individual parts. Finally, the *dynamic* term refers to the fact that the system changes over time depending on its state at some point in time.

In nonlinear systems, the relationship between cause and effect is not proportional and determined, but rather vague and difficult to discern. Nonlinear systems can be characterized by periods of both linear and nonlinear interactions between variables. Thus, dynamic behavior can reveal linear continuity over certain periods of time, while relationships between variables can change, resulting in dramatic structural and behavioral changes in other periods. The dramatic change from a qualitative behavior to another is called "bifurcation."

Chaotic systems are a simple subtype of nonlinear dynamic systems. They can contain very few interactive parts and they can follow very simple rules, but all these systems have a very sensitive dependence on the initial conditions. Despite their deterministic simplicity, these systems can produce a completely unpredictable and chaotic behavior over time. Studies on nonlinear systems highlight three types of temporal behavior: (1) stable behavior (mathematical equilibrium or fixed point); (2) oscillation between mathematical points in a stable, smooth and periodical manner; (3) seemingly random behavior, lacking model (or non-periodic behavior) dominated by uncertainty and in which predictability decomposes. These behaviors may occur intermittently throughout the "life" of a non-linear system. A regime can dominate at certain times, while other regimes dominate at other times [21]. These characteristics determine a variety of behaviors that represent the dynamics of nonlinear systems.

Short history

The roots of the chaos theory date back to 1890, in Henri Poincaré's studies on the problem of the movement of three objects in the mutual gravitational attraction, the so-called three-body problem [17]. In 1898, Jacques Hadamard noted the general divergence of trajectories in negative curvature [10], and Pierre Duhem studied the possible general significance of this in 1908 [3].

In the 1930s, the theory of dynamic systems began to provide characterizations of possible forms of behavior in differential equations. G. D. Birkhoff, A.N. Kolmogorov, M.L. Cartwright, J.E. Littlewood and Stephen Smale have conducted studies on nonlinear differential equations. In the early 1940s, Mary Cartwright and John Littlewood noted that van der Pol’s equation could present sensitive solutions to all figures under its initial conditions.

The theory of chaos progressed more rapidly after the middle of the century, when it became apparent to some scientists that linear theory - the systemic theory that was dominant at that time - simply could not explain the observed behavior of experiments such as the logistic map.

The main catalyst for the development of chaos theory was the electronic computer. A first pioneer of the theory was Edward Lorenz, whose interest in chaos accidentally appeared in his 1961 predictive work [14]. At the end of the 1960s simulations of differential equations with complicated behavior were made, first on analog computers and later on digital computers.

Then, in the mid-1970s, studies of iterative maps with dependence sensitive to initial conditions became commonplace. Robert Shaw's activity at the end of the 1970s clarified the connections between the informational content of the initial conditions and the apparent character of the behavior.

In the early 1980s, indirect signs of chaos were observed in the laboratory in a variety of systems, including electric circuits, lasers, oscillatory chemical reactions, fluid dynamics, and mechanical and magnetomechanical devices, as well as computerized models of chaotic processes. In 1975, the term "chaos" was introduced by T. Li and J. York in scientific discussions and publications, and in December 1977 the Academy of Sciences of New York organized the first symposium on chaos.

In 1982, Mandelbrot published "Fractal Geometry of Nature," which became a classical of the chaos theory ([15]). In 1987, James Gleick published "Chaos: Making a New Science" ([5]), which became a best-seller and introduced the general principles of chaos theory and its history to the general public.

In recent years, major advances have been made in the theory of non-linear dynamic systems, the interest in deterministic chaos has increased enormously, reflected in both literature and real life, and supported by the development of computers.

Description of chaotic behavior

According to Ruelle [18], the first characteristic of the chaotic system is "extreme sensitivity to the initial conditions" and the second, according to James [11], is the existence of "complicated models of nonlinear relationships... which are not really random." In chaotic systems, "minor experiences of individuals can lead to unpredictable changes in the world".

The fundamental properties inherent to the definition of "chaos" are:
(1) non-linear interdependence;
(2) determinism and "hidden" order;
(3) sensitivity to initial conditions.

For any linear system in which the variables are time independent, in the absence of an external drive force, there is a special solution called the **fixed point**. In the case of dissipative systems that are subject to external processes, the notion of fixed point must be generalized to include permanent or repeated movements on a regular basis. These trajectories are called **limit cycles** and, as well as fixed points, may be stable or unstable.

Fixed points and limit cycles are called **attractors**, the trajectories in the state space converging towards them and then remaining very close to them over long periods of time. If a system begins to be observed when it is far from its attractor and is supervised for a long time, it will be possible to detect its movement to the attractor for a while, but from a certain point it will be so close to attractor so that no difference will be noticed. The part of the trajectory along which progress towards the attractor can be observed is called the **transient**. The set of points in the states space on the transients associated with a particular attractor is called the **attractor pool** of the attractor. In a stable linear system, all points in the states space are in the same attraction pool (for any initial system variable configuration, the system's final behavior is the same fixed point or limit cycle).

On the other hand, a nonlinear system can contain more attractors, each with its own attraction pool. Thus, the behavior of a non-linear dynamic system may depend on its initial state, but also on a complete set of phenomena associated with the way the attraction pools change over the variance of the parameters.

In nonlinear systems the differences caused by different initial conditions in this case are not limited to transient effects. This phenomenon of bistability is a generic feature of nonlinear dynamics, showing that nonlinearities play a fundamental role in real systems function.

A generalized feature associated with bistability is **hysteresis** (the dependence of the solution observed by the direction in which a parameter varies). The jump to a different solution in a hysterical system is an example of **bifurcation**. Generally, bifurcation theory describes the transitions that occur between different structural solutions when the system parameter is varied. Such transitions may correspond to the creation or destruction of fixed point elements or simply to changes in the stability properties of existing fixed points.

Nonlinearity can also generate a completely new type of attractor. In nonlinear systems, limit cycles can be quite complicated by cycling in a bounded region of the states space several times before finally closing themselves. It is even possible that a trajectory to be limited to a region of the states space where there are no stable limit cycles or fixed points. The system seems to follow an irregular trajectory that is said to be on a **strange attractor**. The trajectory comes arbitrarily close to its closure itself, but never completely. A strange attractor is the structure of the associated states space specific to the chaotic systems.

In spatial extended systems, non-linearities can generate **pattern**s formation (the spontaneous formation of non-trivial spatial attractors in a system without external inhomogeneities) ([4],[16]).

Elements used in chaos theory

In order to highlight the presence of deterministic chaos in the dynamics of the physical data time series, the following elements are mainly used [1]:

- the power spectrum
- the correlation dimension
- the Lyapunov exponent
- entropy

The power spectrum

For the deduction of the power spectrum, the fast Fourier transform of the time series data is used and power (the average square amplitude) is represented as a function of frequency [22]. The power spectrum is used to qualify the chaotic or quasi-periodic dynamics of the periodic one, and to highlight the periodicities of the processes that determined the time series. A chaotic series is characterized by a broadband spectrum with apparent continuity. The existence of broadband spectrum is only a necessary condition for chaos because quasi-periodic or periodic noise signals can produce the same kind of power spectrum. Therefore, in order to verify the existence of deterministic chaos in a series of time, it is necessary to calculate the correlation dimension and the Lyapunov exponents.

Correlation dimension

In the theory of dynamic systems, it is demonstrated that a *m*-dimensional phases space can be properly reconstructed from a time series, considering the original series$x(t_i)$ and the delayed (deviated) values $x(t_i),\ x(t_i+\tau),\dots,x[t_i+(m-1)\tau]$ as the coordinates of a time series vector:

$$X_i = \{x(t_i),\ x(t_i+\tau),\dots,x[t_i+(m-1)\tau\},$$

where *m* is the size of vector X_i, also called the sinking size of the attractor, and τ is the delay time.

If τ is chosen so that the values $x(t_i)$ are not correlated, then $x(t_i),\ x(t_i+\tau),\dots,x[t_i+(m-1)\tau]$ will be independent, which represents the condition of definition for the phases space.

In calculations, it is usually chosen the value τ of the delay time for which the *auto-correlation function* decreases to $\frac{1}{e}$. Other choices of the value of τ are based on the first value at which the autocorrelation function or its first inflection point is canceled [23].

Having established the phase reconstruction procedure, it is possible to define the correlation integral, according to the Grassberger & Procaccia algorithm [6], as follows:

$$C_m(r) = \mathbf{lim}_{N\to\infty} \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \theta\left(r - |X_i - X_j|\right)$$

where N is the total number of observations, θ is the Heaviside function, and the distance between points is given by the Euclidean norm:

$$|X_i - X_j| = \left[\sum_{k=0}^{m-1} (x_{i+k\tau} - x_{j+k\tau})^2\right]^{1/2}$$

The correlation integal describes the average number of points that are in a volume element of the *m*-dimensional space or the sphere of radius *r* that surrounds each individual state or point on the trajectory $X(t)$ on the attractor.

The correlation dimension is defined by the relationship [22]:

$$d = \lim_{r\to 0}\lim_{N\to\infty}\frac{\log C_m(r)}{\log r}$$

Since the attractor's size is unknown a priori, *d* is calculated for increasing values of *m*. When *d* reaches a value *D* independent of *m*, this value is the attractor's correlation size. The *M* dimension beyond which *d* is no longer varied is the (minimal) sinking magnitude of the attractor. The immediate superior integer value of the correlation dimension *D* indicates the minimum number of independent variables required to describe the evolution in time of the system that generated that time series. $2D+1$ indicates the sufficient number of independent variables for modeling the dynamics of the physical system. A fractional value of the dimension *D* characterizes the chaotic behavior of the dynamic system with dependence sensitive to the initial conditions.

The Lyapunov exponent

The Lyapunov exponent, λ, is a measure of the sensitivity of the dynamic system to the initial conditions. There is a spectrum of Lyapunov exponents; their number is equal to the size of the phase space. If the system is chaotic, there will be at least one positive Lyapunov exponent.

Let (X_j, X_k) be two points in the phase space, corresponding to two different initial conditions, having the distance δ_0 between them at a given time:

$$|X_j - X_k| = \delta_0 << 1$$

After a time corresponding to the number Δn of points in the phase space, the distance between the two considered trajectories becomes:

$$\delta_{\Delta n} = |X_{j+\Delta n} - X_{k+\Delta n}|$$

The maximum Lyapunov exponent is defined by the relationship [12]:

$$\delta_{\Delta n} \cong \delta_0 e^{\lambda \Delta n}, \qquad \delta_{\Delta n} << 1, \qquad \Delta n >> 1$$

If λ is positive, the two trajectories diverge exponentially, which implies the existence of chaos, thus the lack of predictability of the future states of the system.

The maximum Lyapunov exponent determines the predictability of the system as the exponential growth of errors caused by it dims the effects of other exponents equal to zero or negative.

Entropy

The Kolmogorov entropy, *K*, is the most important dimension that can characterize the deterministic chaotic dynamics of a dynamic system. The practical calculation of entropy for a time series of values can be done by two procedures:

- the sum of the positive Lyapunov exponents of an attractor is equal to the dynamic system entropy, $K = \sum \lambda^+$
- if the correlation dimension is saturated, then for *m* large enough ($m > M$) entropy can be calculated using the relationship [24]:

$$K = \frac{1}{n\tau}\ln\frac{C_m(r)}{C_{m+n}(r)}$$

where *r* must be in the linear field of the graph $\log C_m(r)$ as a function of $\log(r)$.

Entropy measures the average rate of time loss of system status information. Entropy inversion provides the timeframe on which a deterministic prediction of a chaotic system can be made. A strange attractor is characterized by a positive value of entropy.

The error *doubling period*, *T*, is defined by the relationship:

$$T = \frac{\ln 2}{K}$$

T is considered the timeframe on which deterministic prediction can be made, beyond which only statistical predictions can be made.

Applications

The theory of chaos is a very vast field, much of it being developed as pure mathematics and not necessarily designed to have practical applications. This category includes fractal art, the "public face" of the theory of chaos. But there have also been practical developments that, unfortunately, remain relatively unknown hidden in the shadow of mathematical evolutions, such as the Mandelbrot sets.

In reality, the theory of chaos has much to offer in the field of practical application. The key to unlocking the power of chaos theory in terms of practical issues is its statistical part. Statistical models developed using the theory of chaos are known as non-linear statistical methods. These methods are particularly powerful because they are qualitative rather than quantitative. Quantitative statistical methods, using such values as standard averages and deviations, do not directly compare all data, but only a small derivative set.

In terms of quantitative methods, it is not unusual for a large collection of data to be reduced to just two numbers as a mean and standard deviation. While this can provide useful information, it loses most of the information in this process. On the other hand, qualitative methods can maintain significantly more information by forming patterns that have data almost as important as the original set.

Applications in experimental sciences

Chaotic dynamics have been observed in a wide variety of experiments in fields such as chemistry, physics, meteorology, hydrology, medicine and biology ([13],[19],[8]). Some of these studies are famous and illustrate the theory. These refer to attractor detection and bifurcation theory with Lyapunov exponent calculation. These works are based on well-known chaotic systems such as the Lorenz, Rössler, Hénon, Chua and Mackey Glass systems. Such models, for example, can explain the behavior of resistance in physics, the water level of a river in hydrology, population growth in ecology, fish flow in fisheries research, or temperature evolution.

Applications in social sciences

Analysis of the individuals behavior also depends on complex models, "closed" models; researchers are interested in knowing their asymptotic behavior. Taking into account, for example, the organization of a market from a social behavior point of view, it is assumed in the theory of balance that the agents have full knowledge of the market. But, in fact, any agent has incomplete knowledge of the market. This knowledge comes from empirical observations, from which agents learn and then make some decisions. Self-regulation can be interpreted in terms of attractor, but no chaotic analytical model corresponds a priori to this. However, this idea has been developed in social sciences based mainly on behavioral investigations.

Applications in economy

Macro-economists have long realized that a certain class of non-linear deterministic systems has been able to produce a self-sustained fluctuation without shocks from outside models. A series of studies has developed numerous examples of deterministic economic models that could generate non-periodic fluctuations. Chaos routes may appear in traditional models of expectation, such as the spider model and asset pricing model by introducing heterogeneous views.

Applications in finance

In [7] it has been shown that some models used in the economy can produce either stable solutions or complex solutions, including a "chaotic" solution. As a result of these ideas, an active research program focused on highlighting chaos as the source of business cycles [20]. In the same way, studies have been developed to understand the dynamics of the labor market or agents' behavior in the markets, using share prices and exchange rates, although attractors' detection is difficult due to the presence of measurement noise [9]. Otherwise, in order to predict data sets whose dynamics seemed complex, attractors' detection was taken into account after deconvolution using wavelets. Chaos in the financial market is discussed in [2].

Conclusions

A chaotic system has three simple features: it is deterministic (it has a determinant equation that governs its behavior); it is sensitive to the initial conditions (even a very small change of the starting point can lead to major changes of the deterministic trajectory); it is neither random nor disordered.

All chaotic systems are nonlinear and involve certain iterative rules. Numerical analysis is usually the only possibility of analyzing such systems. Depending on the initial value of the control parameter, the system can evolve to stable, constant or periodic orbits, or to non-periodic, or chaotic orbits.

In the next phase of the research the focus will be set to identify and investigate the three-dimensional systems Lorentz, Rössler, Chen, Lu, the main algorithms used in the theory of chaos, comparisons between classical deterministic-stochastic methods and methods based on the theory of self-organizing the chaos. A number of applications will be built in order to illustrate the modeling, controlling and synchronization of a chaotic system, including acquisition of signals from the studied process and formation of a time series, analysis of Lyapunov coefficients and their processing, attractors' detection, drawing bifurcation diagrams for the analysis of possibilities to stabilize the system.

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