A new condition for Root Clustering in PMI regions

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Abstract: This paper proposes a new condition for the root clustering of a real matrix A in a complex region $\mathcal{D}$ defined by a polynomial matrix inequality (PMI region). For a general case, a sufficient condition is given so that the eigenvalues of A lie in $\mathcal{D}$. It was shown that this condition is necessary and sufficient for some particular PMI regions including linear matrix inequality (LMI) regions, quadratic matrix inequality (QMI) regions, polynomial regions and many others.

This paper also provides an extension to the classical theorem of Cyparissos Stephanos, which formulates the relationship between the eigenvalues of two matrices and those of composite matrices of their Kronecker products. This extension turned out to be crucial for proving the main results obtained.

Based on the analysis of the proposed condition, a guardian map function can be used for tackling the problem of robust $\mathcal{D}$-stability of single-parameter uncertain linear systems, for which the exact and possibly disconnected domain of $\mathcal{D}$-stability was determined. All the obtained results were illustrated by certain examples.

Keywords: Root clustering, polynomial matrix inequality, PMI region, LMI region, robust $\mathcal{D}$-stability, uncertain systems.

1. Introduction

It is well known that the behavior of a linear dynamical system is closely related to the localization of its eigenvalues in the complex plane (Gantmacher, 2000). This fact explains the interest of the control systems researchers in dealing with the problem of matrix and polynomial root clustering in some particular regions related to different stability types. "Stability" here means having all eigenvalues for a matrix or zeros for a polynomial in the prescribed region.

As this problem is over a century old, several results have been established starting with the most widely known, that is the Routh-Hurwitz criterion (Routh, 1877), that provides a necessary and sufficient condition for determining whether all roots of the characteristic polynomial of a linear system have real parts (Barmish, 1994; Bhattacharyya et al., 1995). Then, a series of important results have also been established (see Kushel (2019) for a detailed history). Among these results especially a general theory of root clustering should be noted, which was developed by Shaul Gutman and Eliahu Ibrahim Jury since the beginning of the 1980's (Gutman & Jury 1981; Gutman, 1984; Gutman, 1990). This theory is based on the notion of transformable Ω-regions and generalized Lyapunov theorem approach. They provided a necessary and sufficient algebraic criterion for the eigenvalues of a matrix to lie in a defined complex region, but the method appears to be very complicate for efficient numerical implementation.

More recently, Chilali & Gahinet (1996) introduced the concept of LMI-regions, which are some regions in the complex plane that can be described by a linear matrix inequality (LMI). They derived a sufficient condition for root clustering in a general class of convex regions, expressed in terms of LMIs. This was a starting point for the emergence of a powerful tool for the analysis and the synthesis of control systems (Duan et al., 2013).

Although many LMI regions are of great importance in control theory, a huge variety of other stability regions cannot be described by LMI. Thus, an extension formulation was introduced in (Peaucelle et al., 2000) under the name of ellipsoidal matrix inequality (EMI) region or quadratic matrix inequality (QMI) region. Most of these regions are convex and connected, which is limitative for separate dynamics systems and many other constraints. To overcome this limitation, Bachelier et al. (2006) proposed a tool for the location of a matrix root in any combination of

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first-order regions based on the $\partial\mathcal{D}$-regularity of a matrix that is the non-intersection of its spectrum with some curve $\partial\mathcal{D}$.

This paper focuses on special regions of the complex plane described by a polynomial matrix inequality (PMI regions). This description includes polynomial regions, LMI regions, QMI regions and many others. A condition is given for root clustering of a matrix A without solving the Lyapunov general equation, just by using the Kronecker product to construct a new matrix depending on A and the matrices used in the definition of the considered region.

The remainder of this paper is organized as follows: Section 2 is dedicated to the $\mathcal{D}$-stability analysis problem. Reminding the reader of the notion of PMI regions, it also presents the main result of the paper, that is a new condition for the root clustering of a real matrix A in a PMI region $\mathcal{D}$. Section 3 tackles the robust $\mathcal{D}$-stability of an uncertain matrix $A(\rho)$. Using the results of Section 2 to define a guardian map that detects when an eigenvalue of $A(\rho)$ reaches the boundary of $\mathcal{D}$, the entire domain to which the uncertain parameter must belongs so that $A(\rho)$ is $\mathcal{D}$-stable is provided. The paper is concluded in Section 4.

2. $\mathcal{D}$-stability analysis problem

2.1. Definition

Let $\mathcal{D}$ be an open sub-region of the complex plane described by the following polynomial matrix inequality:

$$\mathcal{D} = \left\{ z \in \mathbb{C},\ f_{\mathcal{D}}(z) = \sum_{0 \le p,q \le N} Q_{pq} z^p \bar{z}^q < 0 \right\} \tag{1}$$

with $Q_{pq} = Q_{qp}^T \in \mathbb{R}^{mxm}$ any real matrices for any $p,q = 0,1, \ldots, N$.

The region $\mathcal{D}$ is called a PMI region of order N.

A matrix $A \in \mathbb{R}^{nxn}$ is called $\mathcal{D}$-stable if all its eigenvalues lie in $\mathcal{D}$. Note that $f_{\mathcal{D}}(z)$ can be written as follows:

$$f_{\mathcal{D}}(z) = \left(\begin{pmatrix} 1 & z & \cdots & z^N \end{pmatrix} \otimes I_m \right) Q \left(\begin{pmatrix} 1 & z & \cdots & z^N \end{pmatrix} \otimes I_m \right)^H$$

where the exponent H stands for the transpose of the complex conjugate of a matrix and Q is the $m(N+1) \times m(N+1)$ matrix, $Q = \begin{pmatrix} Q_{00} & \cdots & Q_{0N} \\ \vdots & \ddots & \vdots \\ Q_{N0} & \cdots & Q_{NN} \end{pmatrix}$.

Many regions of the complex plane are interesting for system poles confinement for the sake of absolute or relative stability and many other performance specifications. The best-known and most used representation for describing these regions is the LMI description where $f_{\mathcal{D}}(z) = Q_{00} + Q_{10} z + Q_{01} \bar{z}$, with $Q_{10} = Q_{01}^T$. This description initiated by Chilali et al. (1996) was extended by Peaucelle et al. (2000) to define the quadratic matrix inequality (QMI) regions, where, $f_{\mathcal{D}}(z) = Q_{00} + Q_{10} z + Q_{01} \bar{z} + Q_{11} z\bar{z}$, with $Q_{11} \ge 0$.

It is clear that both descriptions are particular cases of the broader PMI description in equation (1). In certain cases, LMI or QMI models cannot represent disconnected and non-convex regions. Hence, there is no analysis method, as far as is known, that can check for $\mathcal{D}$-stability in an efficient way. By contrast, the PMI model can be used to represent such a region very efficiently.

This paper introduces an efficient and attractive algorithm for checking $\mathcal{D}$-stability and even for robust $\mathcal{D}$-stability problems.

2.2. Problem formulation

Let a dynamical system of order n be given by its state space representation:

$$\dot{x}(t) = Ax(t) \tag{2}$$

It will be recalled that the spectrum of the matrix A, noted as σ(A), is the set:

$\sigma(A) = \{\lambda \in \mathbb{C}, \text{ such that } \exists x \neq 0 \in \mathbb{C}^n \text{ with } Ax = \lambda x\}$ of exactly n eigenvalues (counting multiplicities). Equivalently, the eigenvalues of the matrix A are defined as the n roots of its characteristic equation $|A - \lambda I| = 0$.

The following subsections aim to provide an easily testable condition for the matrix A to have all its eigenvalues in the PMI region $\mathcal{D}$.

2.3. PMI region: the general case

For the real matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{l\times l}$ with the eigenvalues $\{\lambda_{i,}, \ i = 1,\ldots,n\}$ and $\{\lambda_{j,}, \ j = 1,\ldots,l\}$ respectively, the matrix *P(A,B)* in $\mathbb{R}^{(nlm)\times(nlm)}$:

$$P(A,B) = \sum_{0\leq p,q\leq N} A^p \otimes B^q \otimes Q_{pq} \tag{3}$$

with $A^0 = I_n$ and $B^0 = I_l$ being the identity matrices of $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{l\times l}$ respectively $Q_{pq} = Q_{qp}^T \in \mathbb{R}^{m\times m}$ and $\otimes$ is the Kronecker product (see Annexe A1). Also, let the matrix $M(x,y)$ be defined in $\mathbb{R}^{m\times m}$ as:

$$M(x,y) = \sum_{0\leq p,q\leq N} x^p y^q Q_{pq} \tag{4}$$

with x and y being any two complex variables. Clearly:

$$M(x,y) = \left(\begin{pmatrix} 1 & x & \cdots & x^N \end{pmatrix} \otimes I_m\right) Q \left(\begin{pmatrix} 1 \\ y \\ \vdots \\ y^N \end{pmatrix} \otimes I_m\right)$$

with Q being the $m(N+1) \times m(N+1)$ matrix: $Q = \begin{pmatrix} Q_{00} & \cdots & Q_{0N} \\ \vdots & \ddots & \vdots \\ Q_{N0} & \cdots & Q_{NN} \end{pmatrix}$

Also, it should be noted that $f_{\mathcal{D}}(z) = M(z,\bar{z})$ for any $z \in \mathbb{C}$.

Lemma 1

The eigenvalues of $P(A,B)$ are those of $M(\lambda_i, \lambda_j)$, when $1 \leq i \leq n$ and $1 \leq j \leq l$.

Proof

Let $A = S_A U_A S_A^H$ (see Annexe A2) where S_A is a $(n\times n)$ unitary matrix ($S_A S_A^H = I_n$) and U_A is a $(n\times n)$ upper triangular matrix with the eigenvalues λ_i of A on the diagonal and let $B = S_B U_B S_B^H$ where S_B is a $(l\times l)$ unitary matrix ($S_B S_B^H = I_l$) and U_B is a $(l\times l)$ upper triangular matrix with the eigenvalues λ_j of B on the diagonal.

Clearly, $A^p = S_A U_A^p S_A^H$ and $B^q = S_B U_B^q S_B^H$ and U_A^p is an upper triangular matrix of dimension $(n \times n)$ with the eigenvalues λ_i^p on the diagonal and U_B^q is an upper triangular matrix of dimension $(l \times l)$ with the eigenvalues λ_j^q on the diagonal. Using the Kronecker product properties (Graham, 1981), the matrix $P(A,B)$ can be written as follows:

$$P(A,B) = \sum_{0 \le p,q \le N} A^p \otimes B^q \otimes Q_{pq}$$

$$= \sum_{0 \le p,q \le N} \left(S_A U_A^p S_A^H\right) \otimes \left(S_B U_B^q S_B^H\right) \otimes \left(I_m Q_{pq} I_m^H\right)$$

$$= \sum_{0 \le p,q \le N} \left(S_A \otimes S_B \otimes I_m\right)\left(U_A^p \otimes U_B^q \otimes Q_{pq}\right)\left(S_A \otimes S_B \otimes I_m\right)^H$$

$$= \sum_{0 \le p,q \le N} T\left(U_{pq} \otimes Q_{pq}\right) T^H$$

with $T = S_A \otimes S_B \otimes I_m$ being, obviously, a unitary matrix of dimension $(mnl \times mnl)$ and $U_{pq} = U_A^p \otimes U_B^q$ being an upper triangular matrix of dimension $(nl \times nl)$ with the eigenvalues $\lambda_i^p \lambda_j^q$ on its diagonal. This finally leads to:

$$P(A,B) = T\left(\sum_{0 \le p,q \le N} U_{pq} \otimes Q_{pq}\right) T^H$$

This expression is clearly a Schur decomposition (Watkins, 2008) of the matrix $P(A,B)$ as T is unitary and $U_H = \sum_{0 \le p,q \le N} U_{pq} \otimes Q_{pq}$ is an upper triangular block matrix with the same dimension as *T*. As a result, $P(A,B)$ and $\sum_{0 \le p,q \le N} U_{pq} \otimes Q_{pq}$ are in fact similar matrices, that is, they have the same eigenvalues. Clearly, the set of these eigenvalues consists of the eigenvalues of every block on the diagonal of $\sum_{0 \le p,q \le N} U_{pq} \otimes Q_{pq}$, namely:

$$\sum_{0 \le p,q \le N} \lambda_i^p \lambda_j^q Q_{pq}$$

which is precisely $M\left(\lambda_i, \lambda_j\right)$. This completes the proof. ■

It must be pointed out that Lemma 1 represents an extension of the well-known theorem of C. Stephanos (Stephanos, 1900) which states that for the matrices $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{l \times l}$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ and $\mu_1, \cdots, \mu_l$ respectively, the eigenvalues of the matrix $\sum_{0 \le p,q \le N} c_{pq} A^p \otimes B^q$, with c_{pq} being some arbitrary scalars, are exactly the *nl* values $\sum_{0 \le p,q \le N} c_{pq} \lambda_i^p \mu_j^q$, for $i = 1, \ldots, n$ and $j = 1, \ldots, l$ (see Annexe A3). When the matrices Q_{pq} in Lemma 1 are of dimension 1, Stephanos's result is retrieved.

In the remainder of this paper $H(A, \mathcal{D})$ will stand for $P(A,A)$ when the matrices Q_{pq} are those used for the description of the PMI region $\mathcal{D}$ through the function $f_{\mathcal{D}}(z) = \sum_{0 \le p,q \le N} Q_{pq} z^p \bar{z}^q$.

That is:

$$H(A, \mathcal{D}) = \sum_{0 \le p,q \le N} A^p \otimes A^q \otimes Q_{pq} \tag{5}$$

Theorem 1

Let $H(A, \mathcal{D})$ be the matrix defined by equation (5). If the real eigenvalues of $H(A, \mathcal{D})$ are all negative, then the roots of *A* lie in the PMI region $\mathcal{D}$.

If $H(A, \mathcal{D})$ has no real eigenvalue then *A* is not $\mathcal{D}$-stable.

Proof

As the PMI region $\mathcal{D}$ is described by $f_{\mathcal{D}}(z) = \sum_{0 \le p,q \le N} Q_{pq} z^p \bar{z}^q < 0$, the eigenvalues of $f_{\mathcal{D}}(z)$ must all be negative, when *z* lies well inside of $\mathcal{D}$. When *z* takes on the eigenvalues of a matrix *A*, the eigenvalues of $f_{\mathcal{D}}(z)$ must all be negative for the matrix *A* to be $\mathcal{D}$-stable. As these eigenvalues coincide with those of $M(\lambda_i, \bar{\lambda}_i)$, when $1 \le i \le n$, which, in turn and by virtue of Lemma 1, are among the real eigenvalues of $H(A, \mathcal{D})$, a sufficient condition for a matrix *A* to be $\mathcal{D}$-stable is that all the real eigenvalues of $H(A, \mathcal{D})$ be negative.

Clearly, on the other hand, if $H(A, \mathcal{D})$ has no real eigenvalues, no eigenvalue *z* of *A* exists such that $f_{\mathcal{D}}(z) < 0$ and, as a result, the matrix *A* cannot be $\mathcal{D}$-stable. ■

Example 1:

Let the PMI region $\mathcal{D}$ be the intersection of a cardioid and a pear form depicted in Figure 1 where **Re** and **Im** stand for real axis and imaginary axis respectively.

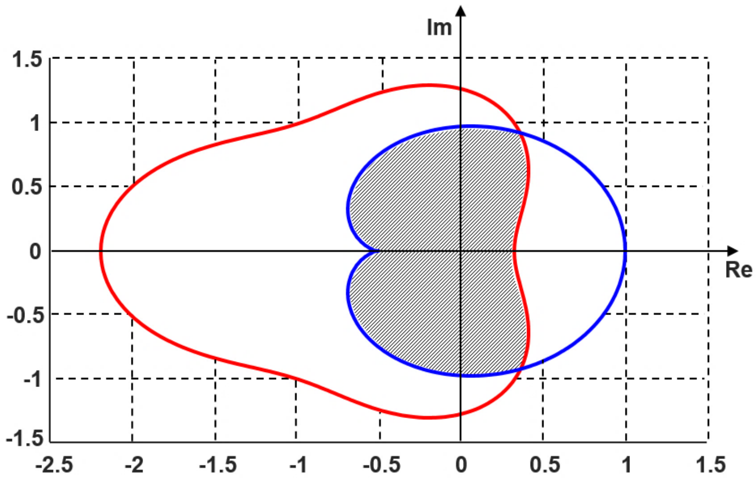

Figure 1. Connected PMI region (dashed)

This region is described by $f_{\mathcal{D}}(z)$ as in equation (1) with:

$$Q_{00} = \begin{pmatrix} -0.125 & 0 \\ 0 & -0.247 \end{pmatrix},\ Q_{01} = \begin{pmatrix} -0.3125 & 0 \\ 0 & -0.169 \end{pmatrix},\ Q_{02} = \begin{pmatrix} 0.0156 & 0 \\ 0 & 1.0375 \end{pmatrix},\ Q_{03} = \begin{pmatrix} 0 & 0 \\ 0 & 0.4 \end{pmatrix},$$

$Q_{11}=\begin{pmatrix}-0.7813 & 0\\ 0 & -0.1375\end{pmatrix}, Q_{12}=\begin{pmatrix}0.25 & 0\\ 0 & 1.6\end{pmatrix}, Q_{13}=\begin{pmatrix}0 & 0\\ 0 & 0.15\end{pmatrix}, Q_{22}=\begin{pmatrix}1 & 0\\ 0 & 1.45\end{pmatrix}, Q_{23}=\begin{pmatrix}0 & 0\\ 0 & 0.3\end{pmatrix}$,

$Q_{33}=\begin{pmatrix}0 & 0\\ 0 & 0.2\end{pmatrix}, Q_{10}=Q_{01}^{T}, Q_{20}=Q_{02}^{T}, Q_{30}=Q_{03}^{T}, Q_{21}=Q_{12}^{T}, Q_{31}=Q_{13}^{T}$ and $Q_{32}=Q_{23}^{T}$.

Let A be a matrix of the form: $A=\begin{pmatrix}1 & -1 & 0\\ 2.5 & -2 & 0\\ 0 & 0 & 0.2\end{pmatrix}$ with eigenvalues $\lambda_{1,2}=-0.5\pm j0.5$ and $\lambda_3=0.2$ which are all in the PMI region. The real eigenvalues of $H(A, \mathcal{D})$ are -0.2744, -0.4344 and -0.0781. As they are all negative, it can be concluded that A is $\mathcal{D}$-stable which is indeed the case.

It must be pointed out that the condition in Theorem 1 is only sufficient, that is the matrix A can be $\mathcal{D}$-stable even if the real eigenvalues of $H(A, \mathcal{D})$ are not all negative. For example, the matrix $A=\begin{pmatrix}-0.4 & 0\\ 0 & 0.3\end{pmatrix}$ is $\mathcal{D}$-stable whereas the real eigenvalues of $H(A, \mathcal{D})$ are -0.3585 -0.05 -0.0184, -0.0014, $\mathbf{0.0213}$ and $\mathbf{0.0659}$. The last two of these values are clearly positive.

Corollary 1

If an eigenvalue of A lies on the frontier of the PMI region $\mathcal{D}$ defined by equation (1), then $|H(A, \mathcal{D})|=0$.

Proof

The PMI region $\mathcal{D}$ is described by $f_{\mathcal{D}}(z)$ as in equation (1). If there exists an eigenvalue λ_i of A on the frontier of $\mathcal{D}$, then $|f_{\mathcal{D}}(\lambda_i)|=0$, which implies that the matrix $M(\lambda_i,\bar{\lambda}_i)$ have one nil eigenvalue. According to Lemma 1 this implies that $|H(A, \mathcal{D})|=0$. ■

2.4. $\mathcal{D}$-stability in a class of PMI regions

A class of PMI regions is considered where the complex domain $\mathcal{D}$ is defined as in equation (1) and it is assumed that the following matrix Q_r is positive semi-definite:

$$Q_r=\begin{pmatrix}Q_{11} & \cdots & Q_{1N}\\ \vdots & \ddots & \vdots\\ Q_{N1} & \cdots & Q_{NN}\end{pmatrix}\geq 0 \tag{6}$$

Lemma 2

For all λ_i,λ_j in the PMI region $\mathcal{D}$ defined by equation (1) and $Q_r\geq 0$, the matrix $\left(M(\lambda_i,\lambda_j)+M^H(\lambda_i,\lambda_j)\right)$ is negative definite.

Proof

$$M(\lambda_i,\lambda_j)=Q_{00}+\sum_{1\leq p,q\leq N}\lambda_i^p Q_{p0}+\lambda_j^q Q_{0q}+\lambda_i^p\lambda_j^q Q_{pq}$$

$$M^H(\lambda_i,\lambda_j)=Q_{00}^T+\sum_{1\leq p,q\leq N}\bar{\lambda}_i^p Q_{p0}^T+\bar{\lambda}_j^q Q_{0q}^T+\bar{\lambda}_i^p\bar{\lambda}_j^q Q_{pq}^T$$

$$M^H\left(\lambda_i,\lambda_j\right)=Q_{00}+\sum_{1\le p,q\le N}\bar{\lambda}_i^p Q_{0p}+\bar{\lambda}_j^q Q_{q0}+\bar{\lambda}_i^p\bar{\lambda}_j^q Q_{qp}$$

The places of *p* and *q* can be switched to obtain:

$$M^H\left(\lambda_i,\lambda_j\right)=Q_{00}+\sum_{1\le p,q\le N}\bar{\lambda}_i^q Q_{0q}+\bar{\lambda}_j^p Q_{p0}+\bar{\lambda}_i^q\bar{\lambda}_j^p Q_{pq}$$

$$M\left(\lambda_i,\lambda_j\right)+M^H\left(\lambda_i,\lambda_j\right)=2Q_{00}+\sum_{1\le p,q\le N}\lambda_i^p Q_{p0}+\lambda_j^q Q_{0q}+\lambda_i^p\lambda_j^q Q_{pq}+\sum_{1\le p,q\le N}\bar{\lambda}_i^q Q_{0q}+\bar{\lambda}_j^p Q_{p0}+\bar{\lambda}_i^q\bar{\lambda}_j^p Q_{pq}$$

It can be written that:

$$\lambda_i^p\lambda_j^q+\bar{\lambda}_i^q\bar{\lambda}_j^p=\left(\lambda_i^p\lambda_j^q-\lambda_i^p\bar{\lambda}_i^q+\bar{\lambda}_i^q\bar{\lambda}_j^p-\bar{\lambda}_j^p\lambda_j^q\right)+\left(\lambda_i^p\bar{\lambda}_i^q+\bar{\lambda}_j^p\lambda_j^q\right)$$
$$=-\left(\lambda_i^p-\bar{\lambda}_j^p\right)\left(\overline{\lambda_i^q-\bar{\lambda}_j^q}\right)+\left(\lambda_i^p\bar{\lambda}_i^q+\bar{\lambda}_j^p\lambda_j^q\right)$$

Hence:

$$M\left(\lambda_i,\bar{\lambda}_j\right)+M^H\left(\lambda_i,\bar{\lambda}_j\right)=f_{\mathcal{D}}\left(\lambda_i\right)+f_{\mathcal{D}}\left(\bar{\lambda}_j\right)-\Lambda_{ij}^H Q_r\Lambda_{ij}$$

Where Λ_{ij} is the $(Nm)\times m$ matrix $\begin{pmatrix}\left(\lambda_i-\bar{\lambda}_j\right)\\ \left(\lambda_i^2-\bar{\lambda}_j^2\right)\\ \vdots\\ \left(\lambda_i^N-\bar{\lambda}_j^N\right)\end{pmatrix}\otimes I_m$.

Now, because λ_i and $\bar{\lambda}_j$ are both located well inside the PMI region $\mathcal{D}$, $f_{\mathcal{D}}\left(\lambda_i\right)$ and $f_{\mathcal{D}}\left(\bar{\lambda}_j\right)$ are both negative definite and because $Q_r\ge 0$, the matrix $M\left(\lambda_i,\bar{\lambda}_j\right)+M^H\left(\lambda_i,\bar{\lambda}_j\right)$ is indeed negative definite. This completes the proof. ■

Theorem 2 gives a necessary and sufficient condition for a matrix *A* to be $\mathcal{D}$-stable with respect to a particular class of PMI region $\mathcal{D}$.

Theorem 2

Let $\mathcal{D}$ be a PMI region with $Q_r\ge 0$. A matrix *A* is $\mathcal{D}$-stable with respect to $\mathcal{D}$ if and only if the real eigenvalues of $H\left(A,\mathcal{D}\right)$ are all negative.

Proof

The sufficiency of the condition has previously been proved by Theorem 1. To prove the necessity, suppose that there exists a positive real eigenvalue μ for $H\left(A,\mathcal{D}\right)$. Then, by Lemma 1, there exist λ_i and λ_j eigenvalues of *A* in the PMI region $\mathcal{D}$ such that μ is also an eigenvalue of $M\left(\lambda_i,\lambda_j\right)$. That is, there exists a non-nil vector $v\in\mathbb{C}^m$ such that $M\left(\lambda_i,\lambda_j\right)v=\mu v$ and $v^H M^H\left(\lambda_i,\lambda_j\right)=\mu v^H$. It then easily follows that $v^H\left(M\left(\lambda_i,\lambda_j\right)+M^H\left(\lambda_i,\lambda_j\right)\right)v=2\mu v^H v$. As $\mu>0$, $M\left(\lambda_i,\bar{\lambda}_j\right)+M^H\left(\lambda_i,\bar{\lambda}_j\right)$ cannot be negative definite. This contradicts the result of Lemma 2.

It is concluded that μ cannot be positive. This completes the proof. ■

Corollary 2

Let D be a PMI region with $Q_r\ge 0$. If $\left|H\left(A,\mathcal{D}\right)\right|=0$, then the matrix *A* is not $\mathcal{D}$-stable.

Proof

If $\left|H\left(A, \mathcal{D}\right)\right| = 0$, then $H\left(A, \mathcal{D}\right)$ has at least one nil eigenvalue. By virtue of Theorem 2, it can be concluded that the matrix A is not $\mathcal{D}$-stable. ■

Example 2: This case is related to a non-convex connected PMI region

Let the PMI region $\mathcal{D}$ be of the form illustrated in Figure 2. This region is described by $f_{\mathcal{D}}\left(z\right)$ as in equation (1) with:

$Q_{00} = 0.1014,\ Q_{01} = Q_{10}^{T} = 130.547,\ Q_{02} = Q_{20}^{T} = 16.318,\ Q_{11} = 128,\ Q_{12} = Q_{21}^{T} = 16$ and $Q_{22} = 2$.

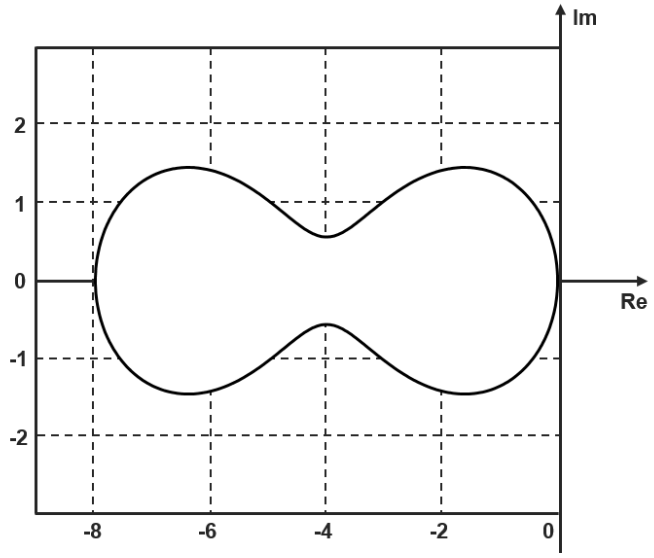

Figure 2. Nonconvex connected PMI region

The matrix $Q_r = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} 128 & 16 \\ 16 & 2 \end{pmatrix}$ is positive semi-definite (the eigenvalues of Q_r are 0 and 130).

- Let A be a matrix of the form: $A = \begin{pmatrix} -3 & -0.5 & 0 \\ 0.5 & -3 & 0 \\ 0 & 0 & -4.5 \end{pmatrix}$ with eigenvalues $\lambda_{1,2} = -3 \pm j0.5$ and $\lambda_3 = -4.5$, which are all in the PMI region $\mathcal{D}$. The real eigenvalues of $H\left(A, \mathcal{D}\right)$ are -30.485 and -17.803. It is clear that, as the eigenvalues of A are in the PMI-region, the matrix $H\left(A, \mathcal{D}\right)$ has only negative eigenvalues.

- Let A be a matrix of the form: $A = \begin{pmatrix} -4.954635 & -1 & 0 \\ 1 & -4.954635 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ with eigenvalues $\lambda_{1,2} = -4.954635 \pm j$ on the frontier of the PMI region $\mathcal{D}$ and $\lambda_3 = -1$ inside $\mathcal{D}$. The real eigenvalues of $H\left(A, \mathcal{D}\right)$ are -130.35 and $\mathbf{0}$.

- It is clear that, as a pair of eigenvalues of A are on the border of the PMI region, the matrix $H\left(A, \mathcal{D}\right)$ has one nil eigenvalue.

- Let A be a matrix of the form: $A=\begin{pmatrix}-1 & -2 & 0\\ 2 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$ with eigenvalues $\lambda_{1,2}=-1\pm 2j$ and $\lambda_3=1$ which are all outside the PMI region $\mathcal{D}$. The real eigenvalues of $H(A,\ \mathcal{D})$ are 171.09 and 455.83.

It is clear that, as all the eigenvalues of A are out of the PMI region, the matrix $H(A,\ \mathcal{D})$ has only positive eigenvalues.

Example 3: This case is related to a disconnected PMI region

Let the PMI region $\mathcal{D}$ be the dashed form in Figure 3.

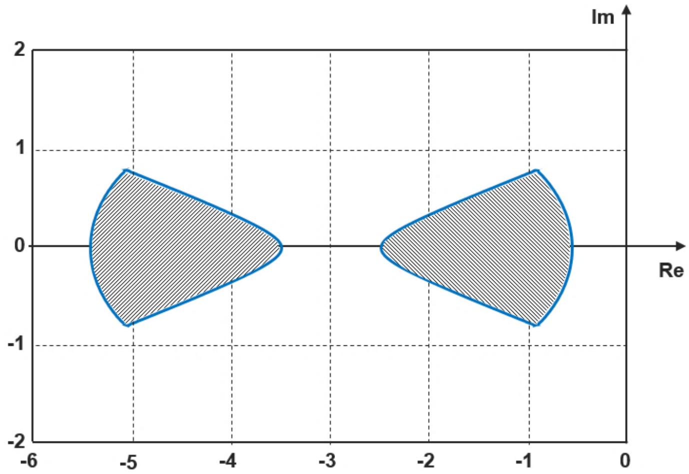

Figure 3. Disconnected PMI region

This region is described by $f_{\mathcal{D}}(z)$ as in equation (1) with:

$$Q_{00}=\begin{pmatrix}-34.19 & 0\\ 0 & 0.81\end{pmatrix},\ Q_{01}=Q_{10}^T=\begin{pmatrix}-11.46 & 0\\ 0 & 0.8733\end{pmatrix},\ Q_{02}=Q_{20}^T=\begin{pmatrix}-7.16 & 0\\ 0 & 0.0067\end{pmatrix},$$

$$Q_{11}=\begin{pmatrix}10.86 & 0\\ 0 & 0.6378\end{pmatrix},\ Q_{12}=Q_{21}^T=\begin{pmatrix}0.06 & 0\\ 0 & 0.06\end{pmatrix}\text{ and }Q_{22}=\begin{pmatrix}0.01 & 0\\ 0 & 0.01\end{pmatrix}.$$

$$Q_r=\begin{pmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}=\begin{pmatrix}10.86 & 0 & 0.06 & 0\\ 0 & 0.6378 & 0 & 0.06\\ 0.06 & 0 & 0.01 & 0\\ 0 & 0.06 & 0 & 0.01\end{pmatrix}$$ is positive definite. The eigenvalues of Q_r are 0.0043, 0.0097, 0.6435 and 10.8603.

- Let A be a matrix of the form: $A=\begin{pmatrix}-5 & -0.5 & 0\\ 0.5 & -5 & 0\\ 0 & 0 & -1\end{pmatrix}$ with eigenvalues $\lambda_{1,2}=-5\pm j0.5$ and $\lambda_3=-1$, which are all in the PMI region $\mathcal{D}$. The real eigenvalues of $H(A,\ \mathcal{D})$ are -14.84, -8.5694, -0.3956 and -0.2638.

It is clear that, as the eigenvalues of A are in the PMI region, the matrix $H(A,\ \mathcal{D})$ has only negative eigenvalues.

- Let A be a matrix of the form: $A=\begin{pmatrix} -5 & -0.5 & 0 \\ 0.5 & -5 & 0 \\ 0 & 0 & -2.5 \end{pmatrix}$ with eigenvalues $\lambda_{1,2}=-5\pm j0.5$ inside the PMI region D and $\lambda_3=-2.5$ on the frontier of the PMI region $\mathcal{D}$. The real eigenvalues of $H(A,\ \mathcal{D})$ are $-8.569,\ -0.971,\ -0.264$ and $\mathbf{0}$.

It is clear that, as one eigenvalue of A is on the border of the PMI region, the matrix $H(A,\ \mathcal{D})$ has one nil eigenvalue.

3. Robust $\mathcal{D}$-stability of an uncertain matrix

This section tackles the asymptotic stability of LTI uncertain systems, which depend on a single real parameter of the form:

$$\dot{x}(t)=A(\rho)x(t),\quad A(\rho)=A_0+\rho A_1 \tag{7}$$

where $A_0,\ A_1\in\mathbb{R}^{n\times n}$ and $\rho\in\ \Omega\subset\mathbb{R}$.

The stability of the systems of the form rendered in equation (7) has long attracted the attention of many researchers within the framework of linear time invariant (LTI) uncertain systems. A summary of the obtained results is given in (Zrida & Bouazizi, 2020) where authors crowned the previous research works by an interesting result that provides the exact robust stability domain for LTI systems, which polynomially depend on a single scalar parameter. Later, the same authors extended this result for robust $\mathcal{D}$-stability in LMI regions (Zrida & Bouazizi, 2022).

Hereafter the same path is followed to tackle the $\mathcal{D}$-stability of systems expressed as in equation (7) in a PMI region $\mathcal{D}$. The following proposition formulates the polynomial dependence of $H(A(\rho),\mathcal{D})$ on the uncertainty parameter ρ.

Proposition

Let $H(A(\rho),\mathcal{D})$ be the matrix defined by equation (3) with $A(\rho)=A_0+\rho A_1$, then:

$$H(A(\rho),\ \mathcal{D})=H_0+\sum_{1\le n\le 2N}\rho^n H_n \tag{8}$$

with:

$$H_0=H(A_0,\mathcal{D}),\qquad H_n=\sum_{\substack{0\le p,q\le N\\ 0\le i\le p\\ 0\le j\le q\\ i+j=n}} w_p(i)\otimes w_q(j)\otimes Q_{pq} \tag{9}$$

where the $(n\times n)$ matrix $w_p(k)$ is the sum of all the words of length p over the alphabet $\{A_0,A_1\}$ where A_1 is repeated exactly k times and $w_0(0)=I_n$.

Proof

The matrix A^p develops to $A^p=(A_0+\rho A_1)^p=\sum_{k=0}^{p}\rho^k w_p(k)$. For example, when p=3, it becomes that $w_3(0)=A_0^3,\ w_3(1)=A_0^2A_1+A_0A_1A_0+A_1A_0^2,\ w_3(2)=A_0A_1^2+A_1A_0A_1+A_1^2A_0$ and $w_3(3)=A_1^3$.

$$A^p \otimes A^q \otimes Q_{pq} = \sum_{0 \le n \le p+q} \rho^n \left(\sum_{\substack{0 \le i \le p \\ 0 \le j \le q \\ i+j=n}}^{p} w_p(i) \otimes w_q(j) \right) \otimes Q_{pq}$$

$$H\left(A(\rho), \mathcal{D}\right) = \sum_{0 \le p,q \le N} \left(\sum_{0 \le n \le p+q} \rho^n \left(\sum_{\substack{0 \le i \le p \\ 0 \le j \le q \\ i+j=n}}^{p} w_p(i) \otimes w_q(j) \right) \right) \otimes Q_{pq}$$

$$H\left(A(\rho), \mathcal{D}\right) = H\left(A_0, \mathcal{D}\right) + \sum_{\substack{0 \le p,q \le N \\ 1 \le n \le p+q}} \rho^n \left(\sum_{\substack{0 \le i \le p \\ 0 \le j \le q \\ i+j=n}}^{p} w_p(i) \otimes w_q(j) \right) \otimes Q_{pq}$$

$$H\left(A(\rho), \mathcal{D}\right) = H\left(A_0, \mathcal{D}\right) + \sum_{1 \le n \le 2N} \rho^n \left(\sum_{\substack{0 \le p,q \le N \\ 0 \le i \le p \\ 0 \le j \le q \\ i+j=n}}^{p} w_p(i) \otimes w_q(j) \otimes Q_{pq} \right)$$

To investigate the $\mathcal{D}$ -stability expressed in equation (7), a guardian map is defined (Saydy et al., 1990; Zhang et al., 2006) for this region by the determinant of $H\left(A(\rho), \mathcal{D}\right)$. By virtue of Corollary1, when an eigenvalue of $A(\rho)$ reaches the boundary of $\mathcal{D}$, $\left|H\left(A(\rho), \mathcal{D}\right)\right| = 0$ is obtained.

Proposition 2 in (Zrida & Bouazizi, 2020) is used. It states that the determinant of the (2N)th-degree pencil $H\left(A(\rho), \mathcal{D}\right)$ coincides with the determinant of the first-degree pencil $H_d + \rho H_c$ with:

$$H_d = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & H_0 \end{pmatrix} \quad \text{and} \quad H_c = \begin{pmatrix} 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ -H_{2N} & -H_{2N-1} & \cdots & -H_2 & -H_1 \end{pmatrix}.$$

For the determination of the complete domain to which ρ must belong so that equation (7) is $\mathcal{D}$ -stable, the main steps of the algorithm of Zrida & Bouazizi (2020) are mentioned below.

- After constructing H_d and H_c by using equation (9), the generalized eigenvalues $\rho_i\left(H_d, H_c\right)$ are determined. For these eigenvalues the determinent $\left|H_d + \rho_i H_c\right|$ is nil, which is equivalent to $\left|H\left(A(\rho), \mathcal{D}\right)\right| = 0$.
- If NaN is a generalized eigenvalue of the pencil $H_d + \rho H_c$, then the system is not $\mathcal{D}$-stable for all ρ values. If not, only the N unduplicated, real and finite values of the generalized eigenvalues $\rho_i\left(H_d, H_c\right)$ are considered for the next step.
- Determine the (N + 1) corresponding open intervals I_i , $i = 1, 2, \cdots$, (N + 1) resulting from the partition of $\mathbb{R}$ produced by these eigenvalues.
- For every open interval I_i, select an arbitrary test point ρ_0 in I_i. If all the real eigenvalues of $H\left(A(\rho_0), \mathcal{D}\right)$ are negative, then $A(\rho_0)$ is $\mathcal{D}$ -stable, and $A(\rho)$ is

$\mathcal{D}$-stable for the entire open interval I_i. If $A(\rho_0)$ is not $\mathcal{D}$-stable, then $A(\rho)$ is not $\mathcal{D}$-stable for the entire open interval I_i.

- Form the exact complete robust stability domain by taking the union of all D-stable open intervals I_i.

Example 4:

a) Consider the PMI region $\mathcal{D}$ in Example 3 and let A_0 and A_1 be the following matrices:

$$A_0 = \begin{pmatrix} -5 & -0.5 & 0 \\ 0.5 & -5 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0.5 & 0 \\ -0.2 & -1 & 0 \\ 0.1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of A_0 are $\lambda_{1,2} = -5 \pm \mathrm{j}0.5$ and $\lambda_3 = -1$, which are all in the PMI region $\mathcal{D}$. For the $\mathcal{D}$-stability of $A_0 + \rho A_1$, it was found that ρ must be in the following intervals:

$$\left]-4.4230,\ -3.6394\right[\cup \left]-2.9105,\ -2.8887\right[\cup \left]-0.6278,\ 0.4256\right[$$

b) Consider the PMI region $\mathcal{D}$ in Example 3 and let A_0 and A_1 be the following matrices:

$$A_0 = \begin{pmatrix} -5 & -0.5 & 0 \\ 0.5 & -5 & 0 \\ 0 & 0 & -2.5 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 1 & 0.8 & 0.4 \\ -0.2 & -1 & -1 \\ 0.5 & 2 & 1 \end{pmatrix}.$$

The eigenvalues of A_0 are $\lambda_{1,2} = -5 \pm \mathrm{j}0.5$ inside the PMI region $\mathcal{D}$ and $\lambda_3 = -2.5$, on the frontier of this region. For the $\mathcal{D}$-stability of $A_0 + \rho A_1$, it was found that ρ must be in the following intervals:

$$\left]-0.6998,\ -0.5865\right[\cup \left]0.0002,\ 0.7243\right[\cup \left]3.1111,\ 3.2598\right[$$

4. Conclusion

This paper provided a new condition for the eigenvalues of a real matrix to lie in a region of the complex plane, which is described by a polynomial matrix inequality. This description not only incorporates LMI, QMI, polynomial regions and others, it also includes regions that are possibly disconnected and/or non-convex. This condition checks for the $\mathcal{D}$-stability in such complicated regions in an easy and an efficient way, without the need of resolving any Lyapunov-type equation, as in (Chilali et al., 1996). Instead, the condition merely consists in computing the eigenvalues of a matrix function, which depends upon the system's matrix and the region description parameters. This matrix happens to provide a guardian map that can be used to tackle the rather hard problem of robust $\mathcal{D}$-stability of single-parameter uncertain systems. These results were only made possible through an important extension provided for the classical theorem of Cyparissos Stephanos.

This paper leads the way into the challenging problem of control synthesis such as pole placement in PMI-regions. This remains an open but quite interesting problem.

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ANNEXES

Three algebraic tools

A1. The Kronecker product

A1.1 Definition

Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product, also known as a direct product or a tensor product of *A* and *B*, denoted by $A \otimes B \in \mathbb{R}^{mp\times nq}$ is defined as the partitioned matrix:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

$A \otimes B$ is seen to be a matrix of order $(mp \times nq)$. It has *mn* blocks, and the $(i, j)^{th}$ block is the matrix $a_{ij}B$ of order $(p \times q)$. It should be noted that $A \otimes B \neq B \otimes A$.

A1.2 Some properties and rules for the Kronecker product

- Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{n\times s}$ and $D \in \mathbb{R}^{q\times t}$ then:
- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- For all *A* and *B*, $(A \otimes B)^T = A^T \otimes B^T$.
- If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are symmetric, then $A \otimes B$ is symmetric.
- If *A* and *B* are nonsingular, then $(A \otimes B)^{-1} = (A)^{-1} \otimes (B)^{-1}$
- If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ are normal, then $A \otimes B$ is normal.
- If $A \in \mathbb{R}^{n\times n}$ is orthogonal and $B \in \mathbb{R}^{m\times m}$ is orthogonal, then $A \otimes B$ is orthogonal.
- Let $A \in \mathbb{R}^{n\times n}$ have a singular value decomposition $A = U_A \Sigma_A V_A^T$ and let $B \in \mathbb{R}^{m\times m}$ have a singular value decomposition $B = U_B \Sigma_B V_B^T$. Then, $(U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A^T \otimes V_B^T)$ yields a singular value decomposition of $A \otimes B$.
- Let $A \in \mathbb{R}^{n\times n}$ have the eigenvalues $\lambda_i, i = 1, 2 \cdots n$ and let $B \in \mathbb{R}^{m\times m}$ have the eigenvalues μ_j, $j = 1, 2 \cdots m$. Then, the *mn* eigenvalues of $A \otimes B$ are $\lambda_1\mu_1, \lambda_1\mu_2, \cdots \lambda_1\mu_m, \lambda_2\mu_1, \lambda_2\mu_2, \cdots \lambda_2\mu_m, \cdots \lambda_n\mu_m$.

Moreover, if x_i, $i=1,2\cdots p$ are linearly independent right eigenvectors of A corresponding to $\lambda_1,\cdots,\lambda_p$, $(p\le n)$, and y_j, $j=1,2\cdots q$ are linearly independent right eigenvectors of B corresponding to $\mu_1,\cdots\mu_q,(q\le m)$, then $x_i\otimes y_j\in\mathbb{R}^{nm}$ are linearly independent right eigenvectors of $A\otimes B$ corresponding to $\lambda_i\mu_j$, $i=1,2\cdots p$, $j=1,2\cdots q$.

A2. Schur Decomposition

A2.1 Existence of the Schur Decomposition

Let $A\in\mathbb{C}^{n\times n}$ be a square matrix. Then there exists a unitary matrix $U\in\mathbb{C}^{n\times n}$ and an upper triangular matrix $T\in\mathbb{C}^{n\times n}$, such that $A=UTU^{-1}$. Moreover, the eigenvalues of A are on the diagonal of T according to their multiplicities.

A2.2 Existence of the Real Schur Decomposition

Let $A\in\mathbb{R}^{n\times n}$ be a square matrix with real eigenvalues. Then, there exists a unitary matrix $U\in\mathbb{R}^{n\times n}$ and an upper triangular matrix $T\in\mathbb{R}^{n\times n}$, such that $A=UTU^{T}$. Moreover, the eigenvalues of A are on the diagonal of T according to their multiplicities.

A3. Stephanos' Theorem

Let $p(x,y)=\sum_{p,q}c_{pq}x^p y^q$, with $c_{pq}\in\mathbb{R}$, be a real polynomial in the two variables x and y and let $P(A,B)=\sum_{p,q}c_{pq}A^p\otimes B^q$ be the associated polynomial of the two matrices $A\in\mathbb{R}^{n\times n}$ with eigenvalues $\lambda_i(A)$ and $B\in\mathbb{R}^{m\mathrm{x}m}$ with eigenvalues $\lambda_j(B)$. The eigenvalues of $P(A,B)$ consist of the nm values $p\big(\lambda_i(A),\lambda_j(B)\big)$ over all possible ordered pairs (i,j), $i=1,\cdots,n \quad j=1,\cdots,m$.

In particular, the eigenvalues of $A\otimes B$ consist of the values of the nm products $\lambda_i(A)\lambda_j(B)$ over all ordered pairs (i,j), $i=1,\cdots,n \quad j=1,\cdots,m$.