Markov processes: branching properties and asymptotic behavior applications in computer science

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Abstract: This paper deals with certain properties of branching Markov processes, in particular, with their asymptotic behaviour, in which context the continuous-time Markov chains are introduced. The main focus of this article is on the asymptotic behaviour of Markov chains, as an original result through which an ergodic property of a Markov chain is transferred to a continuous-time Markov chain induced by a Poisson process. The last part of the article presents concrete applications of continuous-time Markov chains to waiting lines and not only are presented, which are based on the previously mentioned original result.

Keywords: Markov Chains, Branching process, Asymptotic behaviour, Poisson Process, Extension of a Process, Waiting Lines Applications, Expectancy Theory, The Study of Tails, Explosion and Recurrence.

Procese Markov: proprietăți de ramificare și aplicații ale comportamentului asimptotic în informatică

Rezumat: Prezenta lucrare tratează proprietăți ale proceselor Markov de ramificare, în particular, comportamentul lor asimptotic, unde sunt introduse lanțurile Markov în timp continuu. Obiectul central al articolului este cel al comportamentului asimptotic acesta fiind un rezultat original, propriu prin care o proprietate ergodică a unui lanț Markov este tranferată lanțului Markov în timp continuu indus printr-un proces Poisson. În ultima parte a articolului se prezintă aplicații concrete ale lanțurilor Markov în timp continuu la linii de așteptare și nu numai, ce au la bază rezultatul original prezentat anterior.

Cuvinte cheie: Lanțuri Markov, Proces de ramificare, Comportament asimptotic, Proces Poisson, Extinderea unui proces, Aplicații pentru linii de așteptare, Teoria așteptării, Studiul cozilor, Explozie și recurență.

1. Introduction

The name Markov chain comes from the mathematician Andrei Markov.A Markov chain or a Markov process is a stochastic process that has a property according to which its future states are not influenced by its past ones, that is, they are independent from one another. The current state of a Markov process retains all the information about the entire evolution of that process. At any moment of time, a Markov chain can change or keep its state, according to a certain probability distribution.

The Galton-Watson process is a stochastic branching process, which was introduced by Sir Francis Galton in 1889, who studied the evolution of British titles of nobility and other "men of note" families. The possibility of a probabilistic interpretation was formulated precisely in The Educational Time of 1873, issue 4001. This process accurately describes the transmission of the Y chromosome in genetics, and this model is thus useful for understanding Y-chromosome DNA haplogroups. It can be further assumed that the names of the different sons of men are independent random variables, all having the same distribution. As a result, one of the simplest conclusions would be that if the average number of children is one or less, then the child`s last name will disappear, otherwise the probability of survival for a certain number of generations is greater than zero.

Towards the end of the 1930s Leo Szilard reinvented the process for the proliferation of free neurons in a nuclear fission reaction. Modern applications based on this process include the survival probabilities for a new mutant gene or the initiation of a nuclear chain reaction, or the dynamics of disease outbreaks in the first-generation spread or the chances of extinction for small populations of organisms.

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Markov chains are the simplest mathematical models for random phenomena that evolve in discrete time. Their simple structure makes it possible to report results about their asymptotic behavior, the convergence to the equilibrium measure.At the same time, the class of Markov chains is rich enough to serve in many applications. This makes Markov chains the first and foremost example of stochastic processes. The entire mathematical study of random processes can be seen as a generalization in one way or another of the theory of Markov chains.

The remainder of this paper is as follows. Section 2 presents Markov chains, section 3 sets forth Asymptotic behavior for continuous-time Markov chains together with the subsection 3.1 Equilibrium measure for continuous-time Markov chains. Continue with section 4 Waiting lines applications with subsection 4.1 the explosion and recurrence of queues, and finally the conclusion is presented as section 5.

2. Markov chains

Definition of Markov chains:

Let I be a countable set. Each i $\in$ I is called a state and I is called the space of states. It shall be stated that $\lambda = (\lambda_i; i \in I)$ is a measure on I if $0 \le \lambda_i < \infty$ for i $\in$ I.

If in addition $\sum_{i \in I} \lambda_i = 1$, then it can be said that λ is distribution. A probability field $(\Omega, \mathrm{F}, \mathrm{P})$ is established. It will be recalled that a random variable X with values in I is a function $\mathrm{X} : \Omega \to \mathrm{I}$.

If

$$\lambda_i = P(\mathrm{X} = \mathrm{i}) == \mathrm{P}(\omega : \mathrm{X}(\omega) = i), \mathrm{i} \in \mathrm{I}, \text{ then} \tag{1}$$

$\lambda = (\mathrm{p}_i, \mathrm{i} \in \mathrm{I})$ defines the distribution random variables X.

The matrix $P = (\mathrm{p}_{i,j} : \mathrm{i}, \mathrm{j} \in \mathrm{I})$ is stochastic if each line $(\mathrm{p}_{i,j} : \mathrm{i}, \mathrm{j} \in \mathrm{I})$ is a distribution.

The conditions shall be formalised for a Markov chain in terms of the corresponding matrix P. It can be stated that $(\mathrm{X}_n)_{n \ge 0}$ is a Markov chain with initial distribution λ and the matrix of transition P if:

i) X_0 has the distribution λ;

ii) for $n \ge 0$, conditioned by $X_n = i, X_{n+1}$ has distribution $p_{i,j} : j \in I$ and is independent of $X_0, ..., X_{n-1}$.

More explicitly, for $n \ge 0$ and $i_0, ..., i_{n+1} \in I$,

i) $P(\mathrm{X}_0 = \mathrm{i}_0) = \lambda_{i_0}$;

ii) $P(\mathrm{X}_{n+1} = \mathrm{i}_{n+1} | X_0 = \mathrm{i}_0, ..., \mathrm{X}_n = \mathrm{i}_n) = p_{i_n i_{n+1}}$

It shall be stated that $(\mathrm{X}_n)_{n \ge 0}$ is a Markov chain (λ, P).

Observation

Next, to make the following theorem as clear as possible, the concept of martingale theory and the Poisson process shall be explained.

One of the properties of martingales is that they do not take very large or very small values with a probability close to one, being constantly on average. Another property of martingale is related to gambling and represented a betting strategy in France in the 18th century, where the player doubled the stake after each lost game, so that he could recover all his previous losses in the first game he won, plus a win equal to the stake bet. The mathematical model underlying this game is a martingale.

Martingale Convergence Theorem:

Let $(X_n)_{n\in N}$ be a square integrable martingale on the field of probability (Ω, F, P), in relation to the filtration $(F_n)_{n\in N}$.

Then there is a random variable X such that:

$X_n \to X$ almost certainly and in $L^2(P)$

and

$X_n = E(X|F_n)$ for any n.

The Poisson process for the parameter λ:

Let $\lambda > 0$ and $(N_t)_{t\in R_+}$ be a stochastic process on the space (Ω, F, P) with values in N.

$(N_t)_{t\in R_+}$ is called the Poisson process of parameter λ if:

i) $(N_t)_{t\in R_+}$ it is a Levy process, that is, it has cadlag trajectories, $N_0 = 0$ and it has independent and stationary growths;

ii) Process N_t is the Poisson distribution of parameter λ, that is:

$$P \circ N_t^{-1} = e^{-\lambda t} \cdot \sum_{n\geq 0} \frac{(\lambda t)^n}{n!} \cdot \delta_n \text{ for any } t \geq 0. \quad (2)$$

3. Asymptotic behavior for continuous-time Markov chains

A general theory with a Markov chain $(Z_n)_n$, which has a martingale property after convenient normalization was developed, and an ergodic result was derived, i.e. limit at infinity, for the continuous-time Markov chain, $(X_t)_{t\geq 0}$ associated with $(Z_n)_n$:

$X_t := Z_{V_t}$, $t \geq 0$.

Thus, the complete results for the asymptotic behavior were obtained.

Further, the following notations will be used:

Let $X = (X_t, t \geq 0)$ be a Markov chain in continuous time with the transition function $(P_t, t \geq 0)$.

$v_t = (v_t(x), x \in E)$ shall be the distribution of X_t. From the definition of P_t, $v_t = v_0 P_t$ is obtained.

Theorem:

Let $(Z_n)_n$ be a Markov chain with the property that there exists $m > 0$ such that $(\frac{Z_n}{m^n})_n$ is a square integrable martingale. It is assumed that $P(Z_0 > 0) > 0$.

Let $(V_t)_{t\geq 0}$ be a Poisson process of parameter 1 and let $X_t := Z_{V_t}$, $t \geq 0$.

Then, there exists $\lim_{t\to\infty}\frac{X_t}{m^{V_t}}$ almost certainly also in $L^2(\mathrm{P})$.

If this limit is denotes by X:

$$X := \lim_{t\to\infty}\frac{X_t}{m^{V_t}} \tag{3}$$

then X is not zero a.s. that is, $P(\mathrm{X}>0)>0$.

Proof:

It shall be stated that:

$$\lim_n S_n = +\infty \text{ almost certainly} \tag{4}$$

and therefore, the process $(\mathrm{X}_t)_{t\geq 0}$ is well-defined almost certainly.

Let Z be a random variable such that:

$$\lim_n \frac{Z_n}{m^n} = Z \text{ a.c. and in } L^2(\mathrm{P}). \tag{5}$$

Let $\Omega_0 \in F, \Omega_0 \subset \Omega$, such that $P(\Omega \setminus \Omega_0)=0$ and $\lim_n S_n(\omega) = +\infty, \lim_n \frac{Z_n(\omega)}{m^n} = Z(\omega)$, for any $\omega \in \Omega_0$.

Because $Z \in L^2(\mathrm{P})$ it follows that Z is a finite random variable a.c. so $0 < Z < \infty$ a.c. Therefore, it can be assumed that $Z(\omega) < \infty$ for any $\omega \in \Omega_0$.

Let $\omega \in \Omega_0$ and $n_0 \in N$ such that:

$$\left|\mathrm{Z}(\omega) - \frac{Z_n(\omega)}{m^n}\right| < \varepsilon \text{ for any } n \geq n_0. \tag{6}$$

For any $t \geq S_{n_0}(\omega)$ there exists $n = n(\mathrm{t}) \in \mathrm{N}, \mathrm{n} \geq \mathrm{n}_0$, such that: (7)

$V_t(\omega) = \mathrm{n}$

Indeed, this statement follows from the definition of $(\mathrm{V}_t)_{t\geq 0}$. Let $t \geq S_{n_0}(\omega)$, then, using equation (7) the following is obtained:

$$\left|Z(\omega) - \frac{X_t(\omega)}{m^{V_{t(\omega)}}}\right| = \left|Z(\omega) - \frac{Z_{V_t(\omega)}(\omega)}{m^{V_t(\omega)}}\right| = \left|Z(\omega) - \frac{Z_n(\omega)}{m^n}\right| \tag{8}$$

with $n = n(\mathrm{t}) \geq \mathrm{n}_0$. From equation (6) it shall be deduced that:

$$\left|Z(\omega) - \frac{X_t(\omega)}{m^{V_t(\omega)}}\right| < \varepsilon \text{, for any } t \geq S_{n_0}(\omega). \tag{9}$$

It shall be deduced that:

$$\lim_{t\to\infty}\left|Z(\omega) - \frac{X_t(\omega)}{m^{V_t(\omega)}}\right| = 0 \tag{10}$$

or the equivalent equation:

$$\lim_{t\to\infty}\frac{X_t(\omega)}{m^{V_t(\omega)}}=Z_t(\omega)\text{ for any }\omega\in\Omega_0. \tag{11}$$

It can be concluded that:

$$X:=Z=\lim_{t\to\infty}\frac{X_t}{m^{V_t}}\text{ a.c.} \tag{12}$$

Convergence in L^2 remains to be proved.

From the martingale convergence theorem, it follows that $\frac{Z_n}{m^n}$ converges to X in $L^2(\mathrm{P})$.

Let $\varepsilon>0$ then there exists $n_0\in\mathbb{N}$ such that for any $n\geq n_0$:

$$E((Z-\frac{Z_n}{m^n})^2)<\varepsilon \tag{13}$$

the following is obtained:

$$E((Z-\frac{Z_{V_t}}{m^{V_t}})^2)=\sum_{n=0}^{\infty}E((Z-\frac{Z_{V_t}}{m^{V_t}})^2\big|V_t=n)\cdot P(V_t=\mathrm{n})$$

$$=\sum_{n=0}^{\infty}E((Z-\frac{Z_n}{m^n})^2)\cdot\frac{t^n}{n!}\cdot e^{-t}$$

$$=\sum_{n=0}^{n_0-1}E((Z-\frac{Z_n}{m^n})^2)\cdot\frac{t^n}{n!}\cdot e^{-t}+\sum_{n=n_0}^{\infty}E((Z-\frac{Z_n}{m^n})^2)\cdot\frac{t^n}{n!} \tag{14}$$

Since $\lim_{t\to\infty}\frac{t^n}{e^t}=0$ it follows that there exists t_0 such that for any $t\geq t_0$ one obtains $\frac{t^n}{e^t}<\varepsilon$. (15)

Let $M=\max_{0\leq n\leq n_0-1}E((Z-\frac{Z_n}{m^n})^2)$. (16)

From equations (13) and (15) one obtains:

$$E((Z-\frac{Z_{V_t}}{m^{V_t}})^2)<M(1+\frac{1}{1!}+...+\frac{1}{(\mathrm{n}_0-1)!})\cdot\varepsilon+\varepsilon=M(2+\frac{1}{1!}+...+\frac{1}{(\mathrm{n}_0-1)!})\cdot\varepsilon \tag{17}$$

It follows that $\frac{Z_{V_t}}{m^{V_t}}\to Z$ in L^2. (18)

In particular convergence in L^2 implies convergence in L^1.

Then:

$$E(\mathrm{Z})=\lim_{n\to\infty}E(\frac{Z_n}{m^n})=\lim_{n\to\infty}\frac{1}{m^n}E(\mathrm{Z}_n)=\mathrm{E}(\mathrm{Z}_0)\text{, where }P(\mathrm{Z}_0>0)>0. \tag{19}$$

It follows that:

$P(\mathrm{X}>0)>0$. □

Before stating the corollary, the Galton-Watson process was defined.

Let μ be a distribution on N; μ = $\mu = (\mu(\mathrm{k}))_{\mu(\mathrm{k})}$ with $\mu(\mathrm{k}) \geq 0$ and $\sum_{k\geq 0}\mu(\mathrm{k}) = 1$.

Let $Z_{0,}\xi_{n,j}$ $n \in \mathbb{N}, j \in \mathbb{N}^*$ be a family of independent random variables with values in $\mathbb{N}$, so that , $\xi_{n,\mathrm{j}}$ $n \in \mathbb{N}, j \in \mathbb{N}^*$ are identically distributed by the law $\mu; \mathrm{Z}_{0;}\ \xi_{n,j} : (\Omega, \mathrm{F}, \mathrm{P}) \to \mathbb{N}$.

The series of random variables $Z = (\mathrm{Z}_n)_{n\in\mathbb{N}}$ was inductively defined with values in $\mathbb{N}$ as follows:

$$Z_{n+1} := \sum_{j=1}^{Z_n} \xi_{n,j}\ ,\ n \geq 0\,. \tag{20}$$

Let μ_0 be the initial distribution of the process Z, $\mu_0 = P \circ Z_0^{-1}$.

$Z = (\mathrm{Z}_n)_{n\in\mathbb{N}}$ is called the Galton-Watson process with the initial distribution μ_0.

The probability of μ on $\mathbb{N}$ is called the reproduction law $\mu = P \circ \xi_{nj}^{-1}, n \in \mathbb{N}, j \in \mathbb{N}^*$.

Corollary:

Let $(\mathrm{Z}_n)_n$ be a general Galton-Watson process with m > 1 and let $(\mathrm{X}_t)_{t\geq 0}$ be the continuous-time Galton-Watson process generated by $(\mathrm{Z}_n)_n$, the extension of the Markov chain $(\mathrm{Z}_n)_n$ to a Markov process in continuous time. Then, the following exists a.c. and it is in $L^2(\mathrm{P})$:

$$\lim_{t\to\infty}\frac{X_t}{m^{v_t}}$$

If the following limit is denoted by X,

$$\mathrm{X} := \lim_{t\to\infty}\frac{X_t}{m^{v_t}}\ , \tag{21}$$

then X is not zero a.c., i.e. $P(\mathrm{X} > 0) > 0$.

Demonstration:

It was proved that the Galton-Watson process $(\mathrm{Z}_n)_{n\geq 0}$ is a Markov chain and $\frac{Z_n}{m^n}$ is a martingale property, where $m = E(\varepsilon_n)$. Therefore, the main theorem can be applied and the result can be obtained from this statement.

Another rather important concept that was the basis of the proof of the original result is the equilibrium measure for the continuous-time Markov chains.

3.1. The equilibrium measure for the continuous-time Markov chains

Sentence:

Let $X = (\mathrm{X}_n)_n$ be a Markov chain and let us assume that it converges to the equilibrium measure λ:

$$P \circ X_n^{-1} \to \lambda \tag{22}$$

Then λ is the equilibrium measure for the continuous-time Markov chain $(Y_t)_t$ associated with X, where $Y_{t=}X_{V_t}$.

Demonstration:

The generator Q ≔ P – I of the continuous-time Markov chain $(Q_t)_t$, is considered, where P is the transition matrix of X.

It was shown that if $(Q_t)_{t>0}$ is a transition function of $(Y_t)_t$, then $\lambda \circ Q_t = \lambda$, for any t ≥ 0, or it is equivalent to $P^{\lambda} \circ Y_t^{-1} = \lambda$, for any t ≥ 0.

One obtains:
$$Q_t = e^{tQ} = e^{t(P-I)} = e^{-t \cdot e^{tP}} \tag{23}$$

$$\lambda \circ Q_t = e^{-t} \lambda \circ \sum_{m \geq 0} \frac{(tP)^n}{n!} = \lambda \tag{24}$$

because $\lambda \circ P^n = \lambda$, for any n. So λ is also invariant for $(Q_t)_{t\geq 0}$. □

Observation

The result from the corollary above was also obtained in the article: BRANCHING PROCESS AND MARKOV CHAINS IN CONTINUOUS TIME.

4. Applications for waiting lines

The study of the laws of queuing dates back to the results obtained by Agner Krarup Erlang in 1917, who worked for the Copenhagen Telephone Company. Since then, the modeling of queues and networks has evolved considerably due to the diversity and ubiquity of queues:

• ATMs in a large area, counters, transport companies, telephone networks, computer networks, and requests from microprocessors.

The basic models of queues with waiting times and networks are presented, but they are actually representative. We introduce continuous-time Markov chain models are introduced for tails at operator k.

After indicating the infinitesimal generator, one calculated for one server and for k servers, the invariant probability when it exists, the average number of people in queues, and the asymptotic law of the waiting time for a customer entering the system.

Calculating the waiting time law allows one to understand how to scale the number of servers based on the average number of calls per unit time λ, the average call duration 1/μ and a terminal on the average waiting time in the queue. Waiting time can be compared according to several criteria, when the server in the first queue is as efficient as two servers in the second queue.

The connection between certain queues and the Galton-Watson processes is presented in the upcoming sections.

4.1. The explosion and recurrence of queues

A queue is described through a customer arrival process, a model related to how requests are handled.

Arrival time: GI notation is used if the times between successive customer arrivals or inter-arrival times are independent random variables which are governed by the same law. In this case, the arrival process is the counting function associated with the inter-arrival times. If, in addition, the law of arrival times is an exponential law, then the arrival process is a Poisson process, and the notation M will be used to emphasize its Markovian character.

Service times: GI notation is used if the service times are independent random variables which are governed by the same law and independent of the arrival process. If, in addition, the law of arrival times is an exponential law, the notation M will be used.

It will be seen that in this case, if the arrival process is a Poisson process, then the evolution of the queue size can be represented through a Markov chain.

The study of queues with a M/M/1 operator

Similarly, μ can also be interpreted as the average number of people served per unit of time by a server with an infinite number of clients. Thus, traffic density can also be interpreted as the average number of people arriving per unit of time divided by the average number of people served per unit of time.

The explosion and recurrence of M/GI/1 tails

A M/GI/1 queue is considered. The service times $(S_n, n \geq 1)$ for different customers are independent random variables which are governed by the same law.

The process related to the arrival times is a Poisson process with the parameter $\lambda > 0$.

X_t denotes the number of clients in the system at time t. It is assumed that at this time 0, the system has only one client whose service has just started thet is $X_0 = 1$.

This section also includes some significant results that exemplify the previously mentioned notions.

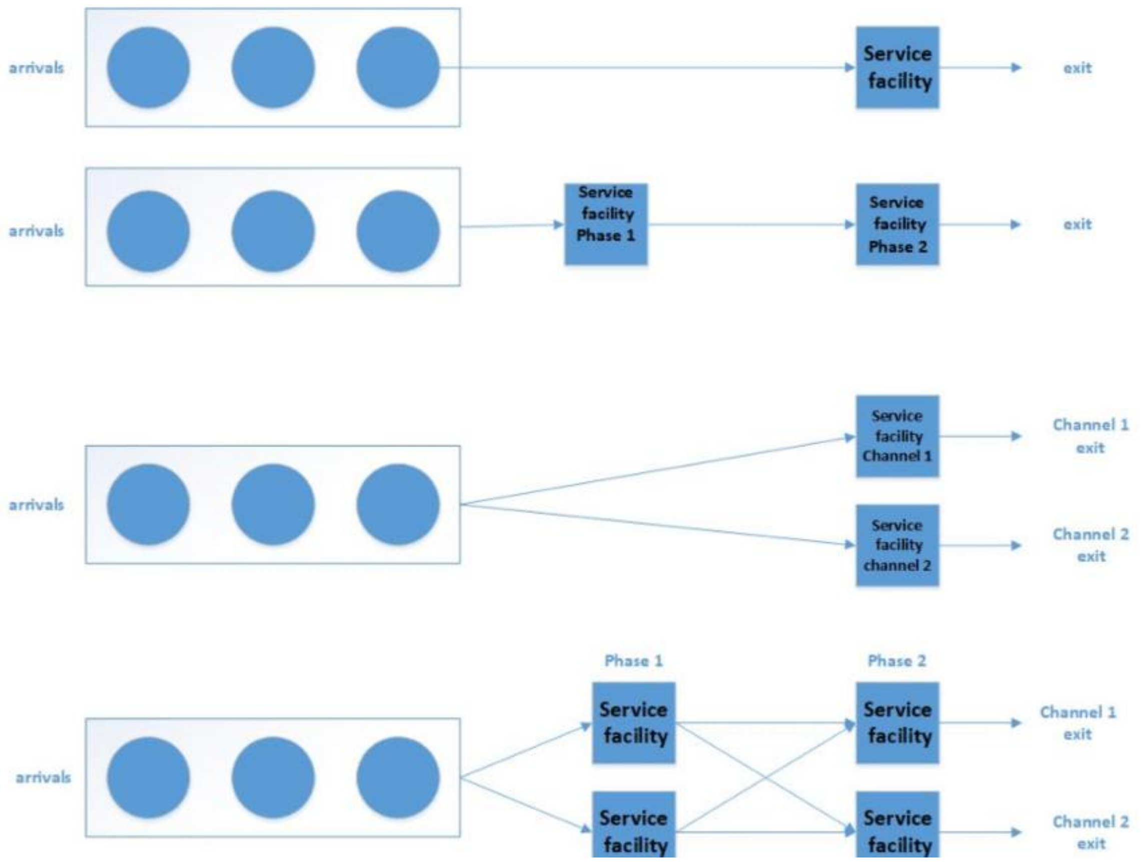

Figure 1. Queuing system design

(Article: Proposed Queuing Model M/M/3 with Infinite Waiting Line; Balkan Journal of applied mathematics and informatics, Managing editor Biljana Zlatanovska; Ph.D. Editor in chief Zoran Zdravev; Ph.D. Technical editor Slave Dimitrov, year 2008, vol. 1, nr. 1)

5. Conclusion

Observation

With the help of the original result, a large number of Galton Watson "genealogical" trees that develop simultaneously can be modeled, and this whole process turns into a Markov process, which helps to optimize the solution development. At the same time, this can be useful for critical phenomena from statistical mechanics, digital forensics, cyber security, etc.

One can model the concept of malware, that encompasses all malicious programs, such as logic bombs, key loggers, Trojan programs, etc. At the same time, there are many approaches to combating these malware programs. The first step in a line of defense is to detect them. An optimal and creative way to detect malware involves modeling the spread of the program as a Markov chain.

Thus, the intruder can be modeled as spreading only according to its current state, regardless of its length (even infinite), because the previously presented theorem helps us to provide a limit to infinity, for a Markov chain in continuous time, by estimating the result. So the detection rate is quite high, namely 90%. In this situation, data mining, as it was defined previously, when combined from the Markov chain perspective, it will go a long way in preventing the spread of malware.

Another example, in the field of Digital Forensics, would be the MC-NPEA model, which is an algorithm based on the notion of the Markov chain (Noise Page Elimination Algorithm for learning, inference, optimization for improving the accuracy of detecting criminal activities).

More precisely, if a machine is in the state y at time n, then the probability that it will go to state x at time n+1 depends only on its current state and not on time n. And so the priority of the user's behavior boils down to a decision on its Markov state. Moreover, to optimize this process it is necessary to combine the Markov chain - the Markov process with the noise removal algorithm. This can be extended to infinity, from where an ergodic result can be deduced.

It is important to mention that there is a tree structure which was created by using the above-mentioned algorithm, which has a root node and weighted edges connecting the rest of the tree nodes. Each forensic action will appear many times as a node in the tree. Weighted paths are used to estimate model transition probabilities.

Therefore, the Markov chain helps to solve and optimize the analysis models for cyber crimes.

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