Robust model predictive control for a class of disturbed systems

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Abstract: This paper proposes a robust model predictive control method for a class of linear discrete time, uncertain and disturbed systems. A relationship between the system disturbance, the states and control input exists, which is used to remove, through several manipulations, the disturbance from the control optimization problem. Moreover, in comparison with other several previous studies, the disturbance does not act directly on the system, a system disturbance matrix being introduced. In principle, the main objective is to find a control law by solving a min-max problem in which a robust performance objective is to be minimized. Instead, an equivalent optimization problem is solved and an upper bound is found for the robust performance objective using a Lyapunov function. With the upper bound, the equivalent control optimization problem is formulated. The solutions of the equivalent optimization problem are used to construct the control law. A Matlab simulation, using Yalmip toolbox, indicates that the states are stabilized to zero and the control input tends to zero.

Keywords: robust model predictive control, linear matrix inequality, uncertain system, disturbance, Schur complement.

Control robust predictiv bazat pe model pentru o clasă de sisteme perturbate

Rezumat: Acest articol propune o metodă de control robust predictiv bazat pe model pentru o clasă de sisteme liniare în timp discret, incerte şi perturbate. Există o legătură între perturbaţia din sistem, stările şi intrarea de control care este utilizată pentru a elimina, prin mai multe manipulări, perturbaţia din problema de optimizare a controlului. Mai mult decât atât, în comparaţie cu alte câteva studii anterioare perturbaţia nu acţionează direct asupra sistemului, fiind introdusă o matrice de perturbaţie a sistemului. În principiu, obiectivul principal este de a găsi o lege de control prin rezolvarea unei probleme min-max în care o funcţie obiectiv de performanţă robustă este de minimizat. În schimb, problema de optimizare echivalentă este rezolvată şi se găseşte o limită superioară pentru funcţia obiectiv de performanţă robustă folosind o funcţie Lyapunov. Cu limita superioară, se formulează problema de control optimal echivalentă. Soluţiile problemei de optimizare echivalente sunt utilizate pentru a construi legea de control. O simulare Matlab, folosind instrumentul Yalmip, indică faptul că stările sunt stabilizate la zero şi intrarea de control tinde spre zero.

Cuvinte cheie: control robust predictiv bazat pe model, inegalitate matricială liniară, sistem incert, perturbaţie, complement Schur.

1. Introduction

Control theory is in charge with creating controllers that influence the behaviour of dynamical systems (Beghdadi, Kouzi, & Ameur, 2023; Fayti et al., 2023; Ibrahim et al., 2023; Kercha et al., 2023). In classical model predictive control (MPC) (Camacho & Bordons, 2007; Rawlings, Mayne & Diehl, 2017; Rădulescu & Ştefănoiu, 2017), a dynamic model is used to predict the future behaviour of the system. The purpose is to minimize a cost function of system performance under both input and output constraints. From the result of the optimization only the first control move is implemented and at the next control step the minimization is executed once more. Robust model predictive control also takes into consideration uncertainties and disturbances (Kothare, Balakrishnan & Morari, 1996; Rădulescu & Ştefănoiu, 2021) and assures that the system is within safe operating limits.

Multiple scientific studies exist on robust model predictive control for systems affected by disturbances. In (Yang et al., 2016) is presented an observer-based output feedback predictive control. The minimal ellipsoidal robust positively invariant set and observer gain are determined through robust positively invariant set conditions of the state estimation error. The authors in (Khan

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et al., 2021) give increased degrees of freedom through a new augmented model with states and tracking error. The control input is determined with a parameter-dependent Lyapunov–Krasovskii functional. Two Model Predictive Control (MPC) methods are considered for the spacecraft at the final phase of the rendezvous maneuver (Mammarella et al., 2018): classical MPC and tube-based robust MPC. In the robust case, an offline feedback gain matrix constitutes the control law and a linear matrix inequality approach is applied to the feedback stabilization criterion. In (Bumroongsri & Kheawhom, 2017), all realizations of the state trajectory are located in offline computed tubes. At each time step, the tubes are included in the optimal control problem. Compared to the online version, the same performance is obtained while the computational time is reduced. The article (Shi et al., 2020) proposes a robust fuzzy predictive control with Takagi-Sugeno model composed of linear submodels and nonlinear membership functions. The state space is augmented with the output tracking error. Stable sufficient conditions are given with the Lyapunov-Krasovskii method and the controller gains are computed based on these conditions. In (Ping & Pedrycz, 2020) is studied output feedback model predictive control for Type-2 Takagi-Sugeno fuzzy systems. The state observer is designed offline and the controller gains for the closed-loop observer system are designed online. The current estimated state is steered from a robust positively invariant set to another one such that future states are invariant in the robust positively invariant set. In (Shi & Mao, 2019) a series of multi-step control sets are computed offline and the convex combination of them is computed online. Input-to-state stability is attained and also robustness to bounded disturbances. The authors consider in (Zhou et al., 2017) RBF-ARX Robust Predictive Control method for tracking without knowing the steady state. The linearized RBF-ARX model that considers the modeling error and bounded uncertain disturbance is used to design the quasi-min–max robust MPC algorithm. Article (Limon et al., 2010) presents robust model predictive control under disturbances for tracking changing targets. If the target changes, the proposed method steers the system to the target if it is admissible. If the target is not reachable, the system is steered to the closest operating point. In (Hu & Ding, 2020) the current control input is computed based on the control from the previous sampling interval. This is one-step ahead control which solves the optimization during the sampling interval, the controller and real system are concurrent. The case of measurable and unmeasurable state is studied. The paper (Kong & Yuan, 2019) presents a disturbance observer and model predictive control method for a nonlinear model subject to disturbances. The fuzzy model predictive control law is designed on the Takagi–Sugeno fuzzy model. In (Ping, Wang & Zhang, 2018) is presented multi-step output feedback robust model predictive control approach for linear parameter varying systems with bounded disturbances. A sequence of controller gains is obtained corresponding to a sequence of Lyapunov matrices and more degrees of freedom for the optimization are introduced. Article (Mayne, Seron & Raković, 2005) considers the initial state of the model to be a decision variable. The disturbance invariant set is the 'origin' when bounded disturbances exist and robust exponential stability of the disturbance invariant set is obtained. The authors in (Yu et al., 2010) present an off-line control law which keeps the trajectories of the error system in a disturbance invariant set. Thus, the evolution of system is in the disturbance invariant set which is centered along the nominal trajectory.

The main contribution of this paper is to determine a robust model predictive control law for a class of disturbed systems. In comparison with (Poursafar, Taghirad & Haeri, 2010), the disturbances do not act directly on the system but through a disturbance system matrix and they have a relationship with the states and control input, so they can be removed from the control optimization problem. A min-max problem is defined where a robust performance objective is to be minimized and an equivalent problem is proposed instead where an upper bound is found on the robust performance objective.

Section 2 presents mathematical preliminaries. Section 3 presents the proposed approach. Section 4 analyzes the simulation results while Section 5 presents the conclusions of the paper.

2. Mathematical preliminaries

Lemma 1 (Schur complement lemma)

Let Ξ be a symmetric matrix of real numbers.

$$\Xi = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \tag{1}$$

If C is positive definite $C > 0$, then:

$$\Xi \geq 0 \Leftrightarrow A - B^T C^{-1} B \geq 0 \tag{2}$$

Lemma 2

Let χ_1, χ_2 be real constant matrices and χ_3 a positive matrix. Then the following holds for any $\varsigma > 0$ (Poursafar, Taghirad & Haeri, 2010):

$$\chi_1^T \chi_3 \chi_2 + \chi_2^T \chi_3 \chi_1 \leq \varsigma \chi_1^T \chi_3 \chi_1 + \varsigma^{-1} \chi_2^T \chi_3 \chi_2 \tag{3}$$

3. Robust model predictive control for a class of disturbed systems

Consider the uncertain discrete time linear system:

$$\begin{aligned} & x(k+1) = A(k)x(k) + B(k)u(k) + G(k)w(k) \\ & \begin{bmatrix} A(k) & B(k) & G(k) \end{bmatrix} \in \Omega \end{aligned} \tag{4}$$

Where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w(k) \in \mathbb{R}^{nw}$, $A(k) \in \mathbb{R}^{n \times n}$, $B(k) \in \mathbb{R}^{n \times m}$, $G(k) \in \mathbb{R}^{n \times nw}$. The polytope Ω, where Co relates to the convex hull is:

$$\Omega = Co\left\{ \begin{bmatrix} A_1 & B_1 & G_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 & G_2 \end{bmatrix}, \ldots, \begin{bmatrix} A_\sigma & B_\sigma & G_\sigma \end{bmatrix} \right\} \tag{5}$$

If $\begin{bmatrix} A(k) & B(k) & G(k) \end{bmatrix} \in \Omega$, then for $\sum_{l=1}^{\sigma} \lambda_l = 1$, $\lambda_l \geq 0$, $l = \overline{1, \sigma}$:

$$\begin{bmatrix} A(k) & B(k) & G(k) \end{bmatrix} = \sum_{l=1}^{\sigma} \lambda_l \begin{bmatrix} A_l & B_l & G_l \end{bmatrix} \tag{6}$$

The disturbance $w(k)$ is in a set $w(k) \in W$:

$$W = \left\{ w(k) \mid w^T(k) w(k) \leq x^T(k) H_1^T H_1 x(k) + u^T(k) H_2^T H_2 u(k) \right\} \tag{7}$$

Where $H_1 \in \mathbb{R}^{n \times n}$, $H_2 \in \mathbb{R}^{m \times m}$.

The proposed control law that stabilizes the system has the following form $u(k) = M(k)x(k)$ and it is norm bounded $u(k) \in U$, $k \geq 0$:

$$U = \left\{ u(k) \mid \left\| u(k) \right\|_2 \leq u_{\max} \right\} \tag{8}$$

Where $u_{\max} \in \mathbb{R}_+^*$.

The min-max problem, with a robust performance objective to be minimized is considered for all $\begin{bmatrix} A(k) & B(k) & G(k) \end{bmatrix} \in \Omega$ and $w(k) \in W$:

$$\min_{u(k+i|k) \in U, i \geq 0} \; \max_{\substack{[A(k+i) \;\; B(k+i) \;\; G(k+i)] \in \Omega \\ w(k+i|k) \in W, i \geq 0}} J_\infty(k) \tag{9}$$

With $J_{\infty}(k)=\sum_{i=0}^{\infty}\left(x^{T}(k+i\mid k)Qx(k+i\mid k)+u^{T}(k+i\mid k)Ru(k+i\mid k)\right)$. The matrices Q and R are positive definite $Q>0$, $R>0$ and $x(k+i\mid k)$, $u(k+i\mid k)$ represent the predicted state and input at $k+i$ from time k.

Consider $V(x(k+i\mid k))=x^{T}(k+i\mid k)P(k)x(k+i\mid k)$ with $P(k)>0$ in order to find an upper bound for the robust performance objective. Let the following inequality hold:

$$V(x(k+i+1\mid k))-V(x(k+i\mid k))\leq-\left(x^{T}(k+i\mid k)Qx(k+i\mid k)+u^{T}(k+i\mid k)Ru(k+i\mid k)\right) \quad (10)$$

Summing (10) from $i=0$ to $i=\infty$:

$$\max_{\substack{[A(k+i)\ \ B(k+i)\ \ G(k+i)]\in\Omega \\ w(k+i|k)\in W,\, i\geq 0}} J_{\infty}(k)\leq V(x(k\mid k)) \quad (11)$$

The upper bound for the robust performance objective is thus found and the problem becomes to minimize $\gamma(k)$ with $V(x(k\mid k))\leq\gamma(k)$ and $V(x(k+i+1\mid k))-V(x(k+i\mid k))\leq$

$$\leq-\left(x^{T}(k+i\mid k)Qx(k+i\mid k)+u^{T}(k+i\mid k)Ru(k+i\mid k)\right)$$

Theorem 1. Let $x(k)=x(k\mid k)$ be the state of the uncertain system (4) at k and $\delta\in\mathbb{R}_{+}^{*}$, $\Lambda\in\mathbb{R}_{+}^{*}$. The robust control law $u(k+i\mid k)=M(k)x(k+i\mid k)$, $\left\|u(k+i\mid k)\right\|_{2}\leq u_{\max}$, $i\geq 0$, $u_{\max}\in\mathbb{R}_{+}^{*}$ is given by $M(k)=Y(k)X^{-1}(k)$, where $X(k)>0$ and $Y(k)$ are the solutions of the optimization problem:

$$\min_{X(k),Y(k),\gamma(k)}\gamma(k)$$

$$s.t$$

$$\begin{bmatrix}1 & x^{T}(k\mid k)\\ x(k\mid k) & X(k)\end{bmatrix}\geq 0$$

$$\begin{bmatrix}\Lambda I_{nw} & G_{l}^{T}\\ G_{l} & X(k)\end{bmatrix}\geq 0, l=\overline{1,\sigma}$$

$$\begin{bmatrix}X(k) & \left(A_{l}X(k)+B_{l}Y(k)\right)^{T} & \overline{H}^{T} & \overline{QR}^{T}\\ A_{l}X(k)+B_{l}Y(k) & \frac{1}{1+\delta}X(k) & 0 & 0\\ \overline{H} & 0 & \overline{\Lambda} & 0\\ \overline{QR} & 0 & 0 & \overline{\gamma}\end{bmatrix}\geq 0, l=\overline{1,\sigma} \quad (12)$$

$$\begin{bmatrix}X(k) & Y^{T}(k)\\ Y(k) & u_{\max}^{2}I_{m}\end{bmatrix}\geq 0$$

With:

$$\overline{H}=\begin{bmatrix}H_{1}X(k)\\ H_{2}Y(k)\end{bmatrix},\ \overline{QR}=\begin{bmatrix}Q^{\frac{1}{2}}X(k)\\ R^{\frac{1}{2}}Y(k)\end{bmatrix},\overline{\Lambda}=diag\left(\frac{1}{\left(1+\delta^{-1}\right)\Lambda}I_{n},\frac{1}{\left(1+\delta^{-1}\right)\Lambda}I_{m}\right), \quad (13)$$

$$\overline{\gamma}=diag\left(\gamma(k)I_{n},\gamma(k)I_{m}\right)$$

Proof:

The first constraint is developed $V(x(k|k)) \le \gamma(k)$.

Substitute $V(x(k|k)) = x^T(k|k)P(k)x(k|k)$ such that $x^T(k|k)P(k)x(k|k) \le \gamma(k)$. Consider $P(k) = \gamma(k)X^{-1}(k)$ such that $x^T(k|k)\gamma(k)X^{-1}(k)x(k|k) \le \gamma(k)$.

Thus $1 - x^T(k|k)X^{-1}(k)x(k|k) \ge 0$. Applying Schur complement lemma leads to:

$$\begin{bmatrix} 1 & x^T(k|k) \\ x(k|k) & X(k) \end{bmatrix} \ge 0 \qquad (14)$$

Next, substitute $x(k+i+1|k)$ from (4) in $V(x(k+i+1|k))$:

$$\begin{aligned}
&V(x(k+i+1|k)) - V(x(k+i|k)) = \\
&= \left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k) + G(k+i)w(k+i|k)\right)^T P(k) \times \\
&\times \left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k) + G(k+i)w(k+i|k)\right) - x^T(k+i|k)P(k)x(k+i|k) = \\
&= \left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right)^T P(k)\left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right) + \\
&+ \left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right)^T P(k)G(k+i)w(k+i|k) + \\
&+ w^T(k+i|k)G^T(k+i)P(k)\left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right) + \\
&+ w^T(k+i|k)G^T(k+i)P(k)G(k+i)w(k+i|k) - x^T(k+i|k)P(k)x(k+i|k)
\end{aligned} \qquad (15)$$

Lemma 2 is used for (15) with $\delta \in \mathbb{R}_+^*$ and the following is obtained:

$$\begin{aligned}
&V(x(k+i+1|k)) - V(x(k+i|k)) \le \\
&\le (1+\delta)\left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right)^T P(k) \times \\
&\times \left(A(k+i)x(k+i|k) + B(k+i)u(k+i|k)\right) + \\
&+ (1+\delta^{-1})w^T(k+i|k)G^T(k+i)P(k)G(k+i)w(k+i|k) - x^T(k+i|k)P(k)x(k+i|k)
\end{aligned} \qquad (16)$$

In order to obtain in (16) $w^T(k+i|k)w(k+i|k)$ instead of $w^T(k+i|k)G^T(k+i)P(k)G(k+i)w(k+i|k)$, a constraint is added to the optimization problem with $\Lambda \in \mathbb{R}_+^*$:

$$G^T(k+i)P(k)G(k+i) \le \Lambda I_{nw}\gamma(k) \qquad (17)$$

If $P(k) = \gamma(k)X^{-1}(k)$, then $G^T(k+i)\gamma(k)X^{-1}(k)G(k+i) \le \Lambda I_{nw}\gamma(k)$. Dividing by $\gamma(k)$ results in $G^T(k+i)X^{-1}(k)G(k+i) \le \Lambda I_{nw}$. Schur complement lemma is applied next:

$$\begin{bmatrix} \Lambda I_{nw} & G^T(k+i) \\ G(k+i) & X(k) \end{bmatrix} \ge 0 \qquad (18)$$

If the following hold for $l = \overline{1,\sigma}$, then (18) holds. These constraints are added to the optimization problem:

$$\begin{bmatrix} \Lambda I_{nw} & G_l^T \\ G_l & X(k) \end{bmatrix} \ge 0, l = \overline{1,\sigma} \qquad (19)$$

Thus, from (17), $G^T(k+i)P(k)G(k+i) - \Lambda I_{nw}\gamma(k) \le 0$ and because it is negative semi-definite, the inequality holds for any $w(k+i|k)$ such that:

$$w^T(k+i|k)\left(G^T(k+i)P(k)G(k+i)-\Lambda I_{nw}\gamma(k)\right)w(k+i|k)\leq 0.$$

So, $w^T(k+i|k)G^T(k+i)P(k)G(k+i)w(k+i|k)\leq \Lambda\gamma(k)w^T(k+i|k)w(k+i|k)$.

In the following, (16) becomes:

$$\begin{aligned}&V(x(k+i+1|k))-V(x(k+i|k))\leq\\&\leq(1+\delta)\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)^T P(k)\times\\&\times\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)+\\&+(1+\delta^{-1})\Lambda\gamma(k)w^T(k+i|k)w(k+i|k)-x^T(k+i|k)P(k)x(k+i|k)\end{aligned}\tag{20}$$

Knowing that:

$$w^T(k+i|k)w(k+i|k)\leq x^T(k+i|k)H_1^TH_1x(k+i|k)+u^T(k+i|k)H_2^TH_2u(k+i|k)$$

leads to:

$$\begin{aligned}&V(x(k+i+1|k))-V(x(k+i|k))\leq\\&\leq(1+\delta)\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)^T P(k)\times\\&\times\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)+\\&+(1+\delta^{-1})\Lambda\gamma(k)\left[x^T(k+i|k)H_1^TH_1x(k+i|k)+u^T(k+i|k)H_2^TH_2u(k+i|k)\right]-\\&-x^T(k+i|k)P(k)x(k+i|k)\end{aligned}\tag{21}$$

If
$V(x(k+i+1|k))-V(x(k+i|k))\leq-\left(x^T(k+i|k)Qx(k+i|k)+u^T(k+i|k)Ru(k+i|k)\right)$ then the following inequality is imposed using (21):

$$\begin{aligned}&(1+\delta)\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)^T P(k)\times\\&\times\left(A(k+i)x(k+i|k)+B(k+i)u(k+i|k)\right)+\\&+(1+\delta^{-1})\Lambda\gamma(k)\left[x^T(k+i|k)H_1^TH_1x(k+i|k)+u^T(k+i|k)H_2^TH_2u(k+i|k)\right]-\\&-x^T(k+i|k)P(k)x(k+i|k)\leq-\left(x^T(k+i|k)Qx(k+i|k)+u^T(k+i|k)Ru(k+i|k)\right)\end{aligned}\tag{22}$$

Substituting $u(k+i|k)=M(k)x(k+i|k)$ leads to the following.

$$\begin{aligned}&(1+\delta)\left(A(k+i)x(k+i|k)+B(k+i)M(k)x(k+i|k)\right)^T P(k)\times\\&\times\left(A(k+i)x(k+i|k)+B(k+i)M(k)x(k+i|k)\right)+\\&+(1+\delta^{-1})\Lambda\gamma(k)\left[x^T(k+i|k)H_1^TH_1x(k+i|k)+x^T(k+i|k)M^T(k)H_2^TH_2M(k)x(k+i|k)\right]-\\&-x^T(k+i|k)P(k)x(k+i|k)\leq\\&\leq-\left(x^T(k+i|k)Qx(k+i|k)+x^T(k+i|k)M^T(k)RM(k)x(k+i|k)\right)\end{aligned}\tag{23}$$

Inequality (23) is equivalent to:

$$\begin{aligned}&(1+\delta)x^T(k+i|k)\left(A(k+i)+B(k+i)M(k)\right)^T P(k)\left(A(k+i)+B(k+i)M(k)\right)x(k+i|k)+\\&+(1+\delta^{-1})\Lambda\gamma(k)x^T(k+i|k)\left[H_1^TH_1+\left(H_2M(k)\right)^T H_2M(k)\right]x(k+i|k)-\\&-x^T(k+i|k)P(k)x(k+i|k)\le\\&\le-\left(x^T(k+i|k)Qx(k+i|k)+x^T(k+i|k)M^T(k)RM(k)x(k+i|k)\right)\end{aligned}\tag{24}$$

Grouping the terms in function of $x(k+i|k)$ leads to:

$$\begin{aligned}&x^T(k+i|k)\left[(1+\delta)\left(A(k+i)+B(k+i)M(k)\right)^T P(k)\left(A(k+i)+B(k+i)M(k)\right)+\right.\\&+(1+\delta^{-1})\Lambda\gamma(k)H_1^TH_1+(1+\delta^{-1})\Lambda\gamma(k)\left(H_2M(k)\right)^T H_2M(k)-P(k)+Q+\\&\left.+M^T(k)RM(k)\right]x(k+i|k)\le 0\end{aligned}\tag{25}$$

If $S\le 0$ then $x^T(k+i|k)Sx(k+i|k)\le 0$, so $x(k+i|k)$ is not considered anymore:

$$\begin{aligned}&(1+\delta)\left(A(k+i)+B(k+i)M(k)\right)^T P(k)\left(A(k+i)+B(k+i)M(k)\right)+(1+\delta^{-1})\Lambda\gamma(k)H_1^TH_1+\\&+(1+\delta^{-1})\Lambda\gamma(k)\left(H_2M(k)\right)^T H_2M(k)-P(k)+Q+M^T(k)RM(k)\le 0\end{aligned}\tag{26}$$

Substitute $P(k)=\gamma(k)X^{-1}(k)$ in (26):

$$\begin{aligned}&(1+\delta)\left(A(k+i)+B(k+i)M(k)\right)^T \gamma(k)X^{-1}(k)\left(A(k+i)+B(k+i)M(k)\right)+\\&+(1+\delta^{-1})\Lambda\gamma(k)H_1^TH_1+(1+\delta^{-1})\Lambda\gamma(k)\left(H_2M(k)\right)^T H_2M(k)-\gamma(k)X^{-1}(k)+Q+\\&+M^T(k)RM(k)\le 0\end{aligned}\tag{27}$$

Divide by $-\gamma(k)$ in (27):

$$\begin{aligned}&-(1+\delta)\left(A(k+i)+B(k+i)M(k)\right)^T X^{-1}(k)\left(A(k+i)+B(k+i)M(k)\right)-(1+\delta^{-1})\Lambda H_1^TH_1-\\&-(1+\delta^{-1})\Lambda\left(H_2M(k)\right)^T H_2M(k)+X^{-1}(k)-\frac{1}{\gamma(k)}Q-\frac{1}{\gamma(k)}M^T(k)RM(k)\ge 0\end{aligned}\tag{28}$$

Multiply on the left with $X^T(k)$ and on the right with $X(k)$ and semi-definiteness is preserved.

$$\begin{aligned}&-(1+\delta)\left(A(k+i)X(k)+B(k+i)M(k)X(k)\right)^T X^{-1}(k)\left(A(k+i)X(k)+B(k+i)M(k)X(k)\right)-\\&-(1+\delta^{-1})\Lambda\left(H_1X(k)\right)^T H_1X(k)-(1+\delta^{-1})\Lambda\left(H_2M(k)X(k)\right)^T H_2M(k)X(k)+\\&+X^T(k)X^{-1}(k)X(k)-\frac{1}{\gamma(k)}X^T(k)QX(k)-\frac{1}{\gamma(k)}\left(M(k)X(k)\right)^T RM(k)X(k)\ge 0\end{aligned}\tag{29}$$

Denote $Y(k)=M(k)X(k)$ such that (29) becomes the following.

$$\begin{aligned}&-(1+\delta)\left(A(k+i)X(k)+B(k+i)Y(k)\right)^T X^{-1}(k)\left(A(k+i)X(k)+B(k+i)Y(k)\right)-\\&-(1+\delta^{-1})\Lambda\left(H_1X(k)\right)^T H_1X(k)-(1+\delta^{-1})\Lambda\left(H_2Y(k)\right)^T H_2Y(k)+\\&+X(k)-\frac{1}{\gamma(k)}X^T(k)QX(k)-\frac{1}{\gamma(k)}Y^T(k)RY(k)\ge 0\end{aligned}\tag{30}$$

If $Q^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are the square roots of Q and R, then:

$$-(1+\delta)\left(A(k+i)X(k)+B(k+i)Y(k)\right)^T X^{-1}(k)\left(A(k+i)X(k)+B(k+i)Y(k)\right)-$$
$$-(1+\delta^{-1})\Lambda\left(H_1X(k)\right)^T H_1X(k)-(1+\delta^{-1})\Lambda\left(H_2Y(k)\right)^T H_2Y(k)+X(k)-$$
$$-\frac{1}{\gamma(k)}\left(Q^{\frac{1}{2}}X(k)\right)^T Q^{\frac{1}{2}}X(k)-\frac{1}{\gamma(k)}\left(R^{\frac{1}{2}}Y(k)\right)^T R^{\frac{1}{2}}Y(k)\geq 0 \qquad (31)$$

Inequality (31) is equivalent to:

$$X(k)-\Psi^T\Upsilon^{-1}\Psi\geq 0 \qquad (32)$$

Where:

$$\Psi=\begin{bmatrix}A(k+i)X(k)+B(k+i)Y(k)\\ \overline{H}\\ \overline{QR}\end{bmatrix},\ \overline{H}=\begin{bmatrix}H_1X(k)\\ H_2Y(k)\end{bmatrix},\ \overline{QR}=\begin{bmatrix}Q^{\frac{1}{2}}X(k)\\ R^{\frac{1}{2}}Y(k)\end{bmatrix}, \qquad (33)$$

$$\Upsilon=diag\left(\frac{1}{1+\delta}X(k),\overline{\Lambda},\overline{\gamma}\right),\ \overline{\Lambda}=diag\left(\frac{1}{(1+\delta^{-1})\Lambda}I_n,\frac{1}{(1+\delta^{-1})\Lambda}I_m\right),$$

$$\overline{\gamma}=diag\left(\gamma(k)I_n,\gamma(k)I_m\right)$$

Schur complement lemma is applied to obtain:

$$\begin{bmatrix}X(k) & \left(A(k+i)X(k)+B(k+i)Y(k)\right)^T & \overline{H}^T & \overline{QR}^T\\ A(k+i)X(k)+B(k+i)Y(k) & \frac{1}{1+\delta}X(k) & 0 & 0\\ \overline{H} & 0 & \overline{\Lambda} & 0\\ \overline{QR} & 0 & 0 & \overline{\gamma}\end{bmatrix}\geq 0 \qquad (34)$$

Because $A(k+i)$ and $B(k+i)$ are time dependent, it is necessary to express the inequality in function of the polytope elements.

$$\begin{bmatrix}X(k) & \left(A_lX(k)+B_lY(k)\right)^T & \overline{H}^T & \overline{QR}^T\\ A_lX(k)+B_lY(k) & \frac{1}{1+\delta}X(k) & 0 & 0\\ \overline{H} & 0 & \overline{\Lambda} & 0\\ \overline{QR} & 0 & 0 & \overline{\gamma}\end{bmatrix}\geq 0, l=\overline{1,\sigma} \qquad (35)$$

If $V(x(k+i+1|k))-V(x(k+i|k))\leq-\left(x^T(k+i|k)Qx(k+i|k)+u^T(k+i|k)Ru(k+i|k)\right)$ and $Q>0$, $R>0$, then $V(x(k+i+1|k))<V(x(k+i|k))$. It is known that $V(x(k|k))\leq\gamma(k)$, so $V(x(k+1|k))<\gamma(k)$. On the same principle $V(x(k+i|k))<\gamma(k)$, $i>0$ and $x^T(k+i|k)X^{-1}(k)x(k+i|k)<1$ because $P(k)=\gamma(k)X^{-1}(k)$. Thus, E is an invariant ellipsoid for the predicted states of the system:

$$E=\left\{\vartheta\mid\vartheta^TX^{-1}(k)\vartheta\leq 1\right\} \qquad (36)$$

The input constraint is the following.

$$\max_{i\geq 0}\left\|u(k+i\,|\,k)\right\|_2^2=\max_{i\geq 0}\left\|Y(k)X^{-1}(k)x(k+i\,|\,k)\right\|_2^2=\max_{i\geq 0}\left\|Y(k)X^{-\frac{1}{2}}(k)X^{-\frac{1}{2}}(k)x(k+i\,|\,k)\right\|_2^2\leq$$
$$\leq\max_{\vartheta\in E}\left\|Y(k)X^{-\frac{1}{2}}(k)X^{-\frac{1}{2}}(k)\vartheta\right\|_2^2\leq\left\|Y(k)X^{-\frac{1}{2}}(k)\right\|_2^2=\lambda_{\max}\left(X^{-\frac{1}{2}}(k)Y^T(k)Y(k)X^{-\frac{1}{2}}(k)\right) \quad (37)$$

The bound is $u_{\max}$, so $\left\|u(k+i\,|\,k)\right\|_2^2\leq u_{\max}^2$ is combined with (37). Impose the constraint:

$$X^{-\frac{1}{2}}(k)Y^T(k)Y(k)X^{-\frac{1}{2}}(k)\leq u_{\max}^2 I_n \quad (38)$$

Multiply on the left and on the right with $X^{\frac{1}{2}}(k)$ such that:

$$Y^T(k)Y(k)\leq u_{\max}^2 X(k) \quad (39)$$

Applying Schur complement lemma results in:

$$\begin{bmatrix} X(k) & Y^T(k) \\ Y(k) & u_{\max}^2 I_m \end{bmatrix}\geq 0 \quad (40)$$

□

4. Numerical results

Matlab with Yalmip toolbox (Lofberg, 2004) is used to implement the presented algorithm in the previous section. The matrices of the polytope are:

$$A_1=\begin{bmatrix} -13.0980 & -5.8441 & 5.7893 \\ -5.8441 & -16.2606 & -5.3504 \\ 5.7893 & -5.3504 & -7.6497 \end{bmatrix},\ A_2=\begin{bmatrix} -13.0980 & -5.8441 & 6 \\ -5.8441 & -16.2606 & -5.3504 \\ 6 & -5.3504 & -7.6497 \end{bmatrix},$$

$$B_1=\begin{bmatrix} -1.2075 & 0 \\ 0.7172 & 1.0347 \\ 1.6302 & 0.7269 \end{bmatrix}, B_2=B_1, G_1=\begin{bmatrix} 0.001 \\ 0.001 \\ 0.001 \end{bmatrix}, G_2=\begin{bmatrix} 0.002 \\ 0.002 \\ 0.002 \end{bmatrix} \quad (41)$$

$$x(k)=\begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^T$$

The disturbance is $w(k)=10^{-1}x_1(k)$. The simulation parameters are:

$$x(0)=\begin{bmatrix} 0.1 & 0.01 & 0.2 \end{bmatrix}^T,\ u_{\max}=10,\ \delta=1,\ \Lambda=10,\ Q=10I_3,\ R=10I_2,$$

$$H_1=\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, H_2=\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad (42)$$

The following Figures (1-5) indicate the simulation results. From Figures 1-3 one can easily see that the states of the system tend to 0. As a result the proposed control approach stabilizes the system. From Figures 4-5 one can easily see that the control input tends to 0. Figure 6 displays the disturbance that acts on the system.

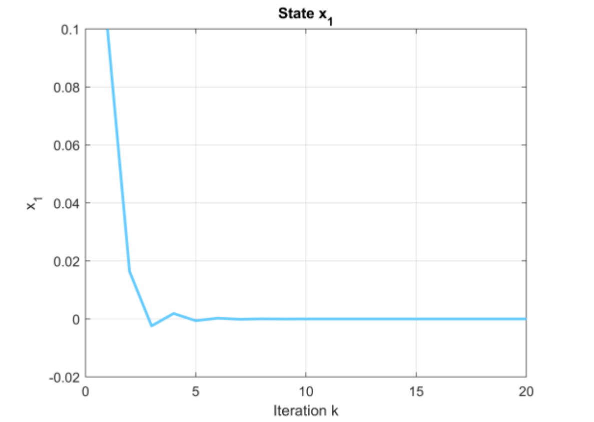

Figure 1. State $x_1(k)$ stabilized

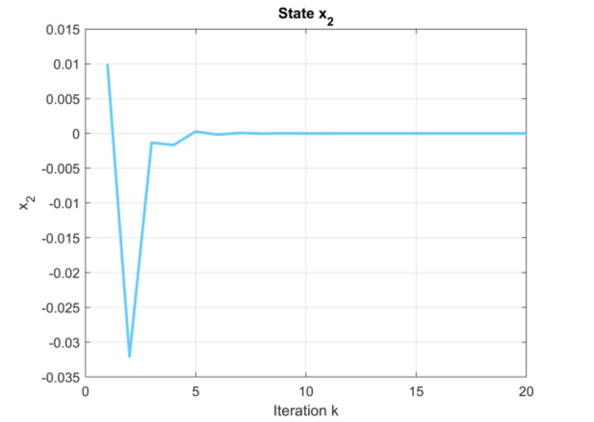

Figure 2. State $x_2(k)$ stabilized

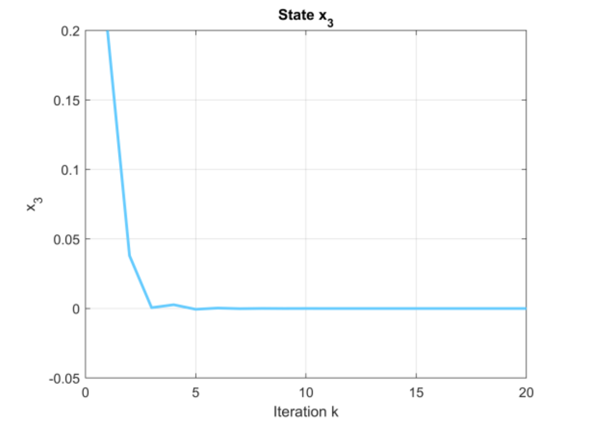

Figure 3. State $x_3(k)$ stabilized

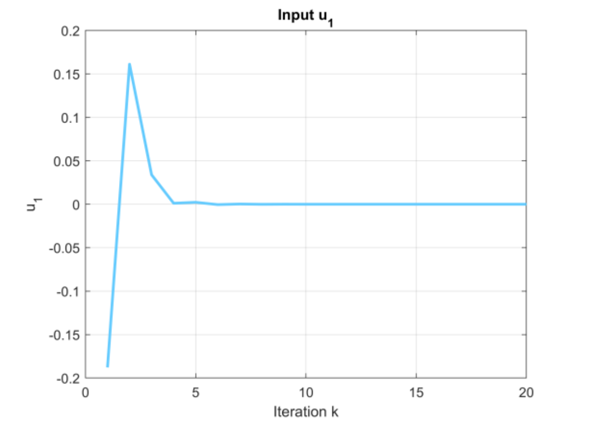

Figure 4. Control input $u_1(k)$

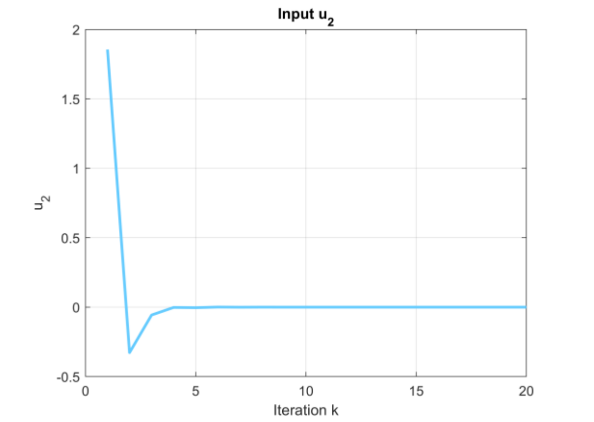

Figure 5. Control input $u_2(k)$

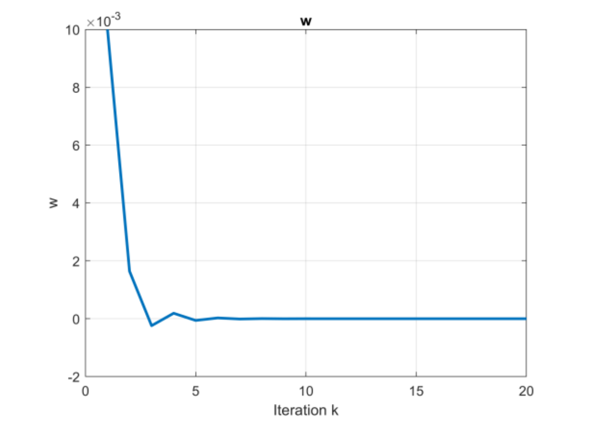

Figure 6. Disturbance $w(k)$

5. Conclusions

Disturbances are uncontrollable influences which affect the output of the system. The result of a disturbance is an increase of the error in the system. An example of a disturbance is the sunlight in a hot summer day in a room which is regulated by an air conditioner. The thermostat has to make now more effort to control the temperature to the desired setpoint.

In this article, a robust model predictive control method for a class of disturbed systems was presented. The aim was to minimize a robust performance objective. In order to solve the optimization problem an upper bound for the robust performance objective was found. An equivalent optimization problem was formulated with the help of this bound. The solutions of this optimization problem were used to construct the control law. Instead of having a disturbance that directly affects the system, a disturbance matrix was introduced. Based on the relationship of the disturbance with the states and control signals, the disturbance was removed from the computations needed to formulate the equivalent problem.

Some examples of potential applications of the proposed robust model predictive control are: an autonomous underwater robot control, a greenhouse temperature control, a trajectory tracking control of robotic manipulators, a three-phase permanent-magnet synchronous motor control etc.

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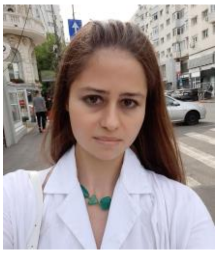

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